

Scalings for a Ballistic Aggregation Equation

Miguel Escobedo · Stéphane Mischler

Received: 6 April 2010 / Accepted: 6 September 2010 / Published online: 16 September 2010
© Springer Science+Business Media, LLC 2010

Abstract We consider a mean field type equation for ballistic aggregation of particles whose density function depends both on the mass and momentum of the particles. For the case of a constant aggregation rate we prove the existence of self-similar solutions and the convergence of more general solutions to them. We are able to estimate the large time decay of some moments of general solutions or to build some new classes of self-similar solutions for several classes of mass and/or momentum dependent rates.

Keywords Aggregation · Ballistic · Smoluchowski equation · Self similar · Mass · Momentum

1 Introduction

The concern of this work is to establish quantitative estimates on the asymptotic behavior of the solutions to some Smoluchowski like models for ballistic aggregation. By ballistic aggregation, also called kinetic coalescence in previous works [7, 13], we mean aggregation phenomena taking place in a system of particles whose density function depends on mass and momentum. It differs from the simplest aggregation mechanism introduced by Smoluchowski [26] in whose model the particles density function only depends on the mass. The close relation between the ballistic aggregation of particles and Smoluchowski type equations is presented in the survey [18].

In order to be more precise let us denote by $P = P_y$ with $y = (m, p)$ a particle of mass $m > 0$ and momentum $p \in \mathbb{R}^d$. The space of particles states is then $Y = \mathbb{R}_+ \times \mathbb{R}^d$ and the velocity of the particle P_y is $v = p/m$. We assume that at a microscopic level (the level of particles) the rate of collision of two particles $P = P_y$ and $P' = P_{y'}$ is a given

M. Escobedo (✉)
Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain
e-mail: mpesmam@lg.ehu.es

S. Mischler
Ceremade, UMR 7534, Université Paris, Dauphine, Place du Maréchal De Lattre de Tassigny,
75775 Paris Cedex 16, France
e-mail: mischler@ceremade.dauphine.fr

nonnegative function $a = a(y, y')$ and when these two particles collide they join to form one aggregated particle $P'' = P_{y''}$ in such a way that the mechanism conserves total mass and total momentum. In other words, the microscopic mechanism reads

$$P_y + P_{y'} \xrightarrow{a(y,y')} P_{y''},$$

with $y'' = (m'', p'')$ given by

$$m'' = m + m', \quad p'' = p + p'.$$

It is worth mentioning that the above reaction dissipates kinetic energy since, denoting $\mathcal{E}^\sharp = m^\sharp |v^\sharp|^2/2$ the kinetic energy of particle P^\sharp , we have

$$\begin{aligned} \mathcal{E}'' - \mathcal{E} - \mathcal{E}' &= \frac{1}{2} \frac{|p + p'|^2}{m + m'} - \frac{1}{2} \frac{|p|^2}{m} - \frac{1}{2} \frac{|p'|^2}{m'} \\ &= -\frac{1}{2} \frac{mm'}{m + m'} |v - v'|^2 \leq 0. \end{aligned}$$

At the mesoscopic (or statistical or mean field) level, the system is described at time $t \geq 0$ by the density function $f(t, y) \geq 0$ of particles with state $y \in Y$. For a given initial distribution f_{in} , the evolution of the density f is described by the Smoluchowski/Boltzmann like equation:

$$\partial_t f = Q(f) \quad \text{in } (0, +\infty) \times Y, \tag{1.1}$$

$$f(0) = f^{in} \quad \text{in } Y. \tag{1.2}$$

The collision operator $Q(f)$ is given by $Q(f) = Q_1(f) - Q_2(f)$, where

$$Q_1(f)(y) = \frac{1}{2} \int_{\mathbb{R}^d} \int_0^m a(y', y - y') f(y') f(y - y') dm' dp', \tag{1.3}$$

$$Q_2(f)(y) = \int_{\mathbb{R}^d} \int_0^\infty a(y, y') f(y) f(y') dm' dp'. \tag{1.4}$$

The form generally accepted of the kernel $a(y, y')$ for ballistic aggregation is $a(y, y') = A(m, m')B(v, v')$ where $A(m, m')$ depends on the sticking probability and the collision cross section and $B(v, v')$ is the collision velocity (see for example [32]). Different precise expressions may be found in the literature. The example $a(m, m') = \gamma_0(m^{1/3} + m'^{1/3})^3$ may be found in [11] for the aggregation of protoplanetary dust. It is obtained considering the collision cross section proportional to the surface of the two colliding particles $(m^{1/3} + m'^{1/3})^2$ and the relative velocity proportional to $m^{1/3} + m'^{1/3}$. In [23] the expression

$$a(y, y') = a_{HS}(y, y') := (m^{1/3} + m'^{1/3})^2 |v - v'|, \tag{1.5}$$

is used in the study of the collapse of rotating dusty protostellar clouds. This second example also appears in the literature of combustion, for example in [31], to describe the aggregation of particles in sprays. Due to the form of its velocity dependence we denote it as a_{HS} where HS stands for hard spheres. A slightly different type of coagulation kernel is considered in [3] and [14] in stellar dynamics and plasma physics, again in an astrophysical context,

where the effects of particle interaction via Newtonian potential on the Vlasov-Boltzmann-Smoluchowski are discussed. The resulting coagulation kernel is

$$a(y, y') = a_{NP}(y, y') := \frac{m + m'}{mm'} \frac{1}{|v - v'|^2}, \tag{1.6}$$

where NP stands for Newtonian potential. We refer to [6, 15, 29, 30] and to the references in [7, 13, 24] for a more detailed discussion on the physics of aggregation.

In the context described above it is very natural to impose that the system has finite initial particle number, total mass and momentum in a certain continuum sense. This reads as the following condition on the initial data f_{in} :

$$0 \leq f^{in} \in L^1(Y, (1 + m + |p|) dy dp). \tag{1.7}$$

Existence of solutions under that condition has been proved in [7, 13, 24]. It has also been proved that

$$f(t, \cdot) \rightarrow 0 \quad \text{in } L^1(Y), \text{ as } t \rightarrow +\infty, \tag{1.8}$$

that is that the total number of particles tends to 0. This is a first result on the long time asymptotic behavior of the solutions but still very partial.

A more detailed description of the asymptotic behavior of the solutions may be obtained by considering scaling properties of (1.1)–(1.4) and the corresponding self similar solutions. Suppose for example that given a solution $f(t, m, p)$ of (1.1)–(1.4), the function $f_r(t, m, p) = r^{-\lambda} f(rt, r^{-\mu}m, r^{-\nu}p)$ is still a solution for any $r > 0$ and for some exponents λ, μ, ν . A self similar solution is then a solution f such that $f = f_r$ for all $r > 0$. It is easy to check that such a function must be of the form $f(t, m, p) = t^\lambda f(1, t^\mu m, t^\nu p)$. These particular solutions may describe sometimes the long time asymptotic behavior of the solutions of the equation for a suitable family of initial data. The existence of such self similar solutions may still be a delicate problem, see for example [9, 12] and the references therein for recent results in that direction for the Smoluchowski equation. We are very far from being able to treat the general case, when the aggregation kernel $a(y, y')$ actually depends on both mass and momentum of the two colliding particles, or even in the case where the aggregation kernel $a(y, y')$ only depends on the momentum of the two colliding particles. We may then be less ambitious and try only to partially improve on the convergence result (1.8). We may imagine to do so in several ways, listed below by order of accuracy. Let us just define before the moment $M_{\bar{\alpha}}(f)$ of order $\bar{\alpha}$ of a function f . It is done, depending on the model considered, as follows:

- when $f = f(y)$ with $y = m \in Y = (0, \infty)$ or $y = p \in Y = \mathbb{R}^d$, then $\bar{\alpha} = \alpha \in \mathbb{R}$ and

$$M_{\bar{\alpha}}(f) = M_{\alpha}(f) = \int_Y |y|^\alpha f dy; \tag{1.9}$$

- when $f = f(y)$ with $y = (m, p) \in Y = (0, \infty) \times \mathbb{R}^d$, then $\bar{\alpha} = (\alpha, \beta) \in \mathbb{R}^2$ and

$$M_{\bar{\alpha}}(f) = M_{\alpha, \beta}(f) = \int_Y m^\alpha |p|^\beta f dy. \tag{1.10}$$

The answers may then be:

- **Answer 1.** Upper bound on moment: $\exists \bar{\alpha}, \exists \nu, C \in (0, \infty)$ such that

$$M_{\bar{\alpha}}(f(t, \cdot)) \leq \frac{C}{t^\nu} \quad \forall t \geq 1.$$

- **Answer 2.** Upper and lower bound on moments: $\exists \bar{\alpha}, \exists v_i = v_i(\bar{\alpha}), C_i = C_i(\bar{\alpha}) \in (0, \infty)$, such that

$$\frac{C_1}{t^{v_1}} \leq M_{\bar{\alpha}}(f(t, \cdot)) \leq \frac{C_2}{t^{v_2}} \quad \forall t \geq 1.$$

- **Answer 3.** Existence of self-similar solution: there exists some profile function $\varphi_\infty : Y \rightarrow \mathbb{R}_+$, some exponents $\lambda, \mu, \nu \in \mathbb{R}$ such that the function

$$\varphi(t, m, p) := t^\lambda \varphi_\infty(t^\mu m, t^\nu p)$$

is a solution to (1.1), (1.3), (1.4).

- **Answer 4.** Self-similar behavior: for any given solution f there exists a self-similar solution φ such that $f \sim \varphi$ as $t \rightarrow \infty$, in a sense to be specified.

Depending on the type of aggregation kernel $a(y, y')$ that we consider, we are able to prove one type of answer or another. The results obtained in this work, still very partial, may be classified as follows.

In Sect. 2 we consider the case of the kernel $a_{HS}(y, y')$ (which depends on both mass and momentum) and the only result that we are able to prove is an upper estimate on some moments (that is a result of type “Answer 1”).

In the remainder of the paper, we focus our attention on easier cases where the aggregation rate a only depends upon the momentum or the mass, namely $a(y, y') = a(p, p')$, $a(y, y') = a(m, m')$ or even $a(y, y') \equiv 1$. In Sect. 3 we consider kernels a only depending on the momentum p and p' . A similar case has been considered in [29]. After integration of the particle density function $f(t, m, p)$ with respect to m , the resulting equation may be seen as describing a set of identical particles moving ballistically and such that when two particles moving with velocities v_1 and v_2 collide they form an aggregate particle moving with velocity $v = v_1 + v_2$. This simplified situation has been considered in [29]. We establish several moment estimates of type “Answer 2” when $a(p, p') = |p - p'|^\gamma$ and deduce that when $\gamma = 2$ (1.1), (1.3), (1.4) has no self-similar solutions of the form described above. This may perhaps suggest also the non existence of self similar solutions in the case of the mass and momentum hard spheres kernel.

Examples where $a \equiv 1$ or $a \equiv a(m, m')$ have already been treated in the literature as simplified models (see for example [23] and [15] for $a = a(m, m')$ and [16] for $a = 1$). When $a = a(m, m')$ it is possible to reduce the original equation (1.1), (1.3), (1.4) to the classical Smoluchowski equation for the zero order moment of the density functions. That type of aggregation rates is sometimes obtained assuming that the velocity v of the particles is determined by their mass m . We treat in Sect. 4 the case where the kernel depends only on the masses m and m' of the colliding particles and we exhibit a new class of self-similar solutions (that is an “Answer 3” type result). Finally, in Sect. 5 the case of constant kernel is treated, for which results of type “Answer 3” and “Answer 4” are established.

The close relation between the ballistic aggregation of particles and Smoluchowski equation, in particular for the additive kernel, has been studied from a different point of view in [1, 2] and references therein. In these works the author shows rigorously the precise relation existing between the deterministic models of ballistic aggregation like sticky particles and the random model of the so-called additive coalescent.

We end this introduction by some remarks and open questions. A common feature of these equations is that

$$M_{1,0}(t) \equiv M_{1,0}(0) \quad \text{and} \quad M_{0,0}(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

and when the cross-section a is homogeneous of order $\bar{\gamma}$ (which belongs to \mathbb{R} or \mathbb{R}^2) it is likely that

$$M_{\bar{\gamma}}(t) \equiv \frac{1}{t} \quad \text{as } t \rightarrow \infty, \tag{1.11}$$

a result which is also known to be true for the coagulation equation (see [9, 10, 12]) and for the inelastic Boltzmann equation (see [20] and the references therein). The identity (1.11) has been established when the aggregation rate depends only on the momentum or on the mass. But only one side of such identity is proved in the case of the true hard spheres aggregation rate, that depends both on the mass and the momentum of the particles. We ask then.

Open question 1 Does the asymptotic identity behavior (1.11) holds true for some mass and momentum depending aggregation rate? Numerical simulations indicate that this may be the case in some of the problems treated in [28] where the authors consider a coagulation kernel depending both on mass and momentum of degree $\bar{\gamma} = 0$. Strong numerical evidences indicate that in dimension $d = 2$ and for a large reduced density of particles the zero order moment $M_0(t)$ behaves like t^{-1} .

Another interesting question should be to establish some asymptotic behavior of typical velocity or momentum depending quantity. A way to express that in mathematical terms is the following:

Open question 2 Is it possible to exhibit some moment $M_{\bar{\alpha}}$ for which we may determinate the long time behavior of $M_{\bar{\alpha}}/M_0$ (even just saying that it converges as $t \rightarrow +\infty$)?

2 Mass and Momentum Dependence Case: a Remark on the Hard Spheres Model

We start recalling an existence result for initial data f_{in} satisfying the symmetry property $f_{in}(p) = f_{in}(-p)$ for all $p \in \mathbb{R}^d$. The functions satisfying that property will be called even functions in all the remaining of this paper.

Theorem 2.1 (cf. [13, Theorem 2.6, Theorem 2.8 and Lemma 3.3]) *Assume that the aggregation rate a satisfies*

$$\begin{aligned} 0 \leq a(y, y') &= a(y', y) \leq k_S(y)k_S(y') \quad \forall y, y' \in Y, \\ a(m, -p, m', -p') &= a(m, p, m', p') \quad \forall (m, p), (m', p') \in Y, \\ a(m, p, m', p') &\leq a(m, p, m', -p') \quad \forall (m, p), (m', p') \in Y \text{ s.t. } \langle p, p' \rangle > 0, \end{aligned}$$

with $k_S(y) := 1 + m + |p| + |v|$. Then, for every non negative and even (in the p variable) initial condition $f_{in} \in L^1(Y; k_S^2(y)dy)$, there exists a unique solution of (1.1)–(1.4) $f \in$

$C([0, T]; L^1(Y; k_S(y)dy)) \cap L^\infty(0, T; L^1(Y; k_S^2(y)dy)) \forall T > 0$, satisfying furthermore:

$$\int_Y f(t, \cdot)m dy \equiv Cst, \tag{2.1}$$

$$f(t, \cdot) \text{ is even, so that } \int_Y f(t, \cdot)p dy \equiv 0, \tag{2.2}$$

$$\int_Y f(t, \cdot)|v|^k dy \leq \int_Y f_{in}|v|^k dy \quad \forall k > 0, \tag{2.3}$$

$$\int_Y f(t, \cdot)|p|^2 dy \leq \int_Y f_{in}|p|^2 dy, \tag{2.4}$$

$$\int_Y fm^\alpha dy \rightarrow 0 \text{ when } t \rightarrow \infty, \forall \alpha < 1. \tag{2.5}$$

Remark 2.2

- (i) It is worth mentioning that the hard spheres collision rate a_{HS} does satisfy the assumption of Theorem 2.1, but not the Manev rate a_{NP} .
- (ii) As a consequence of (2.1), (2.3), (2.4) and (2.5) we deduce that

$$M_{\alpha,\beta}(t) := \int_Y f(t, y)m^\alpha|p|^\beta dy \rightarrow 0 \text{ as } t \rightarrow \infty \tag{2.6}$$

whenever (α, β) belongs to the region

$$\{\beta \in [0, 2], \alpha < 1 - \beta/2\} \cup \{\beta \geq 2, \alpha < 2 - \beta\}.$$

In the case of the hard spheres model we are able to quantify the rate of decay of one of the moment functions of the solution. More precisely, we have the following result.

Lemma 2.3 Assume that $a = a_{HS}$ and the assumption of Theorem 2.1 hold true. Then the solution f of (1.1)–(1.4) is such that $A^{-1} := M_{-1/3,1}(0) < \infty$ and

$$M_{-1/3,1}(t) \leq \frac{1}{A + t/4} \quad \forall t \geq 0. \tag{2.7}$$

Proof Notice first that $M_{-1/3,1}(0) < \infty$ because

$$m^{-1/3}|p| = m^{2/3}|v| \leq m^{4/3} + |v|^2 \leq 2k_S^2.$$

Now, from the expression (1.1)–(1.2) of the collision kernel we have

$$\int_Y Q(f, f)m^{-1/3}|p| dy = \frac{1}{2} \int_Y \int_Y \Delta_{-1/3,1}ff' dy dy',$$

with

$$\Delta_{-1/3,1} = [(m + m')^{-1/3}|p + p'| - m^{-1/3}|p| - (m')^{-1/3}|p'|][r + r']^2|v - v'|.$$

On one hand $-\Delta_{-1/3,1} \geq 0$ because

$$(m + m')^{1/3} \left(\frac{|p|}{m^{1/3}} + \frac{|p'|}{(m')^{1/3}} \right) \geq |p| + |p'| \geq |p + p'|.$$

On the other hand, if we only take into account the values of v and v' where $v \cdot v' < 0$ and suppose that, for example, $|p| = \min(|p|, |p'|)$ we have

$$\begin{aligned}
 -\Delta_{-1/3,1} &\geq \left(\frac{|p|}{m^{1/3}} + \left(\frac{|p'|}{(m')^{1/3}} - \frac{|p'|}{(m+m')^{1/3}} \right) \right) [r^2 + (r')^2][|v| + |v'|] \\
 &\geq \left(\frac{|p|}{m^{1/3}} \right) [(r')^2][|v'|] = \frac{|p|}{m^{1/3}} \frac{|p'|}{(m')^{1/3}}.
 \end{aligned}$$

Whence, using that f is even:

$$\begin{aligned}
 \frac{d}{dt} \int_Y f \frac{|p|}{m^{1/3}} dy &\leq -\frac{1}{2} \int_{Y^2, v \cdot v' < 0} \frac{|p|}{m^{1/3}} \frac{|p'|}{(m')^{1/3}} f f' dy dy' \\
 &\leq -\frac{1}{4} \left(\int_Y f \frac{|p|}{m^{1/3}} dy \right)^2,
 \end{aligned}$$

from which (2.7) straightforwardly follows. □

Remark 2.4 As it has already been noticed (cf. for example [4, 5, 27]), if the aggregation rate is of the form $a(y, y') = A(m, m')B(v, v')$ (as in a_{HS} or a_{NP}) and we assume that $f(t, m, p)$ is a solution of (1.1), (1.3), (1.4) of the form $F(t, m)\varphi(pm^{-\theta})$ for some function φ such that $\int \varphi(p)dp = 1$ and $\theta \in \mathbb{R}$, then $F(t, m)$ satisfies a Smoluchowski equation with a coagulation rate given by $A(m, m')C(m, m')$ with C depending on the function φ and on B :

$$\begin{aligned}
 \frac{\partial F}{\partial t}(t, m) &= \frac{1}{2} \int_0^m F(t, m - m')F(t, m')A(m - m', m')C(m - m', m') dm' \\
 &\quad - \int_0^\infty F(t, m)F(t, m')A(m, m')C(m, m') dm', \tag{2.8}
 \end{aligned}$$

$$C(m, m') = m^{-6\theta} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi\left(\frac{p}{m^\theta}\right) \varphi\left(\frac{p'}{m'^\theta}\right) B\left(\frac{p}{m}, \frac{p'}{m'}\right) dp dp'. \tag{2.9}$$

Conversely, consider any φ for which the kernel $C(m, m')$ is well defined, and F a solution of the Smoluchowski equation (2.8) with coagulation rate $A(m, m')C(m, m')$, where $C(m, m')$ is given by (2.9). Then, the function $f(t, m, p) = \varphi(mp^{-\theta})F(t, m)$ satisfies a kind of averaged (in v) version of (1.1), (1.3), (1.4), namely:

$$\partial_t \int_{\mathbb{R}^3} f(t, m, p) dp = \int_{\mathbb{R}^3} (Q_1(f) - Q_2(f))(t, m, p) dp. \tag{2.10}$$

Of course all the available results on coagulation equation may be applied to (2.8). But the function f does not satisfies (1.1), (1.3), (1.4) unless $B \equiv 1$ since it is not possible to find any function φ such that:

$$m^{-3\theta} \int_{\mathbb{R}^3} \varphi\left(\frac{p''}{m^\theta}\right) B\left(\frac{p''}{m}, \frac{p'}{m'}\right) dp'' = B\left(\frac{p}{m}, \frac{p'}{m'}\right).$$

Notice that, since no uniqueness result of the solutions to the Cauchy problem for (2.10) is known, we cannot even know if the function $F(t, m)$ solving (2.8), (2.9) coincides with the “ v -average” of the solution $f(t, m, p)$ of (1.1), (1.3), (1.4). The case $B \equiv 1$ is treated in Sect. 4.

3 The Momentum Dependence Case $a = a(p, p')$

We consider now (1.1), (1.3), (1.4) with a collision kernel a independent of the mass of the colliding particles. We may then integrate the equation with respect to the mass and obtain that the function of t and p , $\int_0^\infty f(t, m, p) dm$, that we shall still denote f , satisfies the equation:

$$\partial_t f = Q(f, f) \quad \text{in } (0, +\infty) \times \mathbb{R}^d, \tag{3.1}$$

$$f(0) = f_{in} \quad \text{in } \mathbb{R}^d, \tag{3.2}$$

the collision operator $Q(f)$ is given by $Q(f, f) = Q_1(f, f) - Q_2(f, f)$, where

$$Q_1(f, f)(y) = \frac{1}{2} \int_{\mathbb{R}^d} a(p', p - p') f(p') f(p - p') dp', \tag{3.3}$$

$$Q_2(f, f)(p) = \int_{\mathbb{R}^d} a(p, p') f(p) f(p') dp'. \tag{3.4}$$

We focus on the cases

$$a(p, p') = |p - p'|^\gamma, \quad \gamma \in [0, 2], \quad d \in \mathbb{N}^*. \tag{3.5}$$

Before stating our main result we need some definitions and notations. We say that a function f on \mathbb{R}^d is radially symmetric if

$$f(Rp) = f(p) \quad \forall p \in \mathbb{R}^d, \quad R \in SO(d)$$

where $SO(d)$ stands for the group of rotations on \mathbb{R}^d . For any weight function $k : \mathbb{R}^d \rightarrow \mathbb{R}_+$ we define the ‘‘moment of order k ’’ of the non negative density measure $f \in M_{loc}^1(\mathbb{R}^d)$ by

$$M_k(f) := \int_{\mathbb{R}^d} k(p) f(dp),$$

and we define M_k^1 as the set of Radon measures μ such that $M_k(|\mu|) < \infty$. For any $\alpha \in \mathbb{R}_+$ we use the shorthand notation

$$M_\alpha := \int_{\mathbb{R}} f(p) |p|^\alpha dp,$$

that is $M_\alpha = M_k(f)$ for $k(p) = |p|^\alpha$ and the shorthand notation $M_\alpha^1 = M_\ell^1$ for $\ell(p) = 1 + |p|^\alpha$.

Theorem 3.1 Consider the aggregation rate (3.5).

- (i) For any even initial datum $f_{in} \in M_{2\alpha}^1$, $\alpha \in \mathbb{N} \setminus \{0, 1\}$, there exists a unique even solution $f \in C([0, T]; M^1(\mathbb{R}^d) - \text{weak}) \cap L^\infty(0, T; M_{2\alpha}^1(\mathbb{R}^d))$ to (3.1)–(3.4). For any $\alpha \in [0, 1]$ the function $t \mapsto M_\alpha(t)$ is decreasing and $f(t, \cdot)$ is radially symmetric for any $t \geq 0$ if furthermore f_{in} is radially symmetric.
- (ii) Moreover, the solution $f(t, \cdot)$ satisfies

$$\frac{1}{M_\gamma(0)^{-1} + k_1 t} \leq M_\gamma(t) \leq \frac{1}{M_\gamma(0)^{-1} + k_2 t} \quad \forall t \geq 0, \tag{3.6}$$

for some constants $k_i = k_i(\gamma, d) \in (0, \infty)$.

One of the main tools in order to establish that result is to consider the equations satisfied by the moments of the solution f . Using the classical argument for the coagulation equation, and one more change of variable $p' \rightarrow -p'$, it is easy to check that any even solution f to (3.1)–(3.4) satisfies (at least formally) the fundamental moment equation

$$\begin{aligned} \frac{d}{dt} M_\alpha &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f f' a(p, p') [|p + p'|^\alpha - |p|^\alpha - |p'|^\alpha] dp dp' \\ &= \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f f' \{ a(p, p') [|p + p'|^\alpha - |p|^\alpha - |p'|^\alpha] \\ &\quad + a(p, -p') [|p - p'|^\alpha - |p|^\alpha - |p'|^\alpha] \} dp dp'. \end{aligned} \tag{3.7}$$

We consider in this section the case $\gamma \in (0, 2)$ and $d \in \mathbb{N}^*$, the case $\gamma = 1$ and $d = 1$ and the case $\gamma = 2$ and $d \in \mathbb{N}^*$. The case $\gamma = 0$ and $d = 1$ is treated in Sect. 5.

3.1 Proof of the Existence and Uniqueness Part in Theorem 3.1

We prove in this subsection a uniqueness and existence result for a general class of aggregation rates by adapting some arguments from [13, 17], see also [22]. We then deduce the existence and uniqueness part in Theorem 3.1.

Lemma 3.2 *Consider a continuous aggregation rate $a : \mathbb{R}^{2d} \rightarrow \mathbb{R}_+$ which satisfies*

$$a(-p, -p') = a(p, p') \quad \forall p, p' \in \mathbb{R}^d, \tag{3.8}$$

$$a(p, p') \leq a(-p, p') \quad \forall p, p' \in \mathbb{R}^d, p \cdot p' > 0, \tag{3.9}$$

and an even weight function $k : \mathbb{R}^d \rightarrow \mathbb{R}_+$. We define

$$\begin{aligned} \Delta_k(p, p') &:= a(p, p') [k(p'') + k(p') - k(p)], \\ \tilde{\Delta}_k(p, p') &= \Delta_k(p, p') + \Delta_k(-p, p'). \end{aligned}$$

and assume that

$$a(p, p') \leq Ck(p)k(p') \quad \text{and} \quad \tilde{\Delta}_k(p, p') \leq Ck(p)k(p')^2. \tag{3.10}$$

Then, for any given even initial data $f_{in} \in M_k^1(\mathbb{R}^d)$ there exists at most one even solution $f \in C([0, T]; M_k^1(\mathbb{R}^d)) \cap L^\infty(0, T; M_{k,2}^1(\mathbb{R}^d))$ to (3.1)–(3.4).

Remark 3.3

- (i) The same result holds without the assumption that the initial density function f_{in} is even when the second condition in (3.10) is replaced by

$$\Delta_k(p, p') \leq Ck(p)k(p')^2.$$

The same kind of results was obtained in [13, 17] in the L^1 framework. The same result also holds for radially symmetric solutions when we assume that

$$a(Rp, Rp') = a(p, p') \quad \forall p, p' \in \mathbb{R}^d, R \in SO(d), \tag{3.11}$$

and the second condition in (3.10) is replaced by

$$\int_{R \in SO(d)} \Delta(p, Rp') dR \leq Ck(p)k(p')^2.$$

(ii) The same kind of result holds for aggregation rate defined on Y^2 with $Y = (0, \infty) \times \mathbb{R}^d$ as it is the case when particles are identified by their mass and momentum, see [13].

Proof of Lemma 3.2 Step 1. We claim that for any $g_{in} \in M_k^1$ there is a unique $g \in C([0, T]; M_k^1 - weak)$, $G \in L^1(0, T; M_k^1)$ and $b \in C((0, T) \times \mathbb{R}^d; \mathbb{R}_+)$ such that

$$\begin{aligned} \partial_t g &= G - bg \quad \text{in the sense of } \mathcal{D}'([0, T] \times \mathbb{R}^d), \\ g(0) &= g_{in} \quad \text{in } M_k^1. \end{aligned}$$

and that, the differential inequality

$$\frac{d}{dt} \|gk\|_{M^1} \leq \|Gk\|_{M^1} - \|bgk\|_{M^1} \tag{3.12}$$

holds in the sense of $\mathcal{D}'([0, T])$. First, it is clear using a classical duality argument that (3.12) has at most one solution. Suppose indeed, $g_1, g_2 \in C([0, T]; M_k^1 - weak)$ are two such solutions. For any $\bar{t} \in (0, T)$, and any $\bar{\varphi} \in C_0(\mathbb{R}^d)$ the function $\varphi(t, p) = \bar{\varphi}(p) \exp \int_{\bar{t}}^t b(s, p) ds$ satisfies $\varphi(t, \cdot) \in C_0(\mathbb{R}^d)$ for all $t > 0$, solves the dual homogeneous equation $\partial_t \varphi = b\varphi$ in $\mathbb{R} \times \mathbb{R}^d$ and $\varphi(\bar{t}) = \bar{\varphi}$. Let finally be $\psi_n \in C_0([0, T])$ such that $\psi_n(t) = 1$ on $t \in [0, \bar{t}]$, $\psi_n(s) \rightarrow \mathbf{1}_{[0, \bar{t}]}(s)$ for all $s \in [0, T)$ and $\psi'_n \rightarrow -\delta_{\bar{t}}$ as $n \rightarrow +\infty$. Then, on the one hand:

$$\begin{aligned} (\partial_t(g_1 - g_2), \varphi \psi_n) &= - \int_0^T \int_{\mathbb{R}^d} (\psi_n \varphi)_t d(g_2(s) - g_1(s))(p) ds \\ &= - \int_0^T \int_{\mathbb{R}^d} \varphi_t \psi_n d(g_2(s) - g_1(s))(p) ds \\ &\quad - \int_0^T \int_{\mathbb{R}^d} \psi_{n,t} \varphi d(g_2(s) - g_1(s))(p) ds \\ &= - \int_0^T \int_{\mathbb{R}^d} b\varphi \psi_n d(g_2(s) - g_1(s))(p) ds \\ &\quad - \int_0^T \int_{\mathbb{R}^d} \psi_{n,t} \varphi d(g_2(s) - g_1(s))(p) ds. \end{aligned}$$

And on the other hand,

$$\begin{aligned} (\partial_t(g_1 - g_2), \varphi \psi_n) &= -(b(g_1 - g_2), \varphi \psi_n) \\ &= - \int_0^T \int_{\mathbb{R}^d} b\varphi \psi_n d(g_2(s) - g_1(s))(p). \end{aligned}$$

We deduce that, for all $n \geq 1$:

$$\int_0^T \int_{\mathbb{R}^d} \psi_{n,t} \varphi d(g_2(s) - g_1(s))(p) ds = 0$$

and passing to the limit $n \rightarrow +\infty$, using that $g_i \in C([0, T], M_k^1 - weak)$:

$$\int_{\mathbb{R}^d} \bar{\varphi} d(g_2(t) - g_1(t))(p) = 0.$$

In order to show the existence of a solution $g \in C([0, T]; M_k^1 - weak)$ of (3.12) we notice first that, for any $g_\varepsilon(0) \in C_{K_\varepsilon} := \{u \in C(\mathbb{R}^d); \text{supp}u \subset K_\varepsilon\}$, with $K_\varepsilon \subset \mathbb{R}^d$ a compact, and any $G_\varepsilon \in L^1(0, T; C_{K_\varepsilon})$ there exists a (unique) solution $g_\varepsilon \in C([0, T]; C_{K_\varepsilon})$ to (3.12) which furthermore satisfies

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |g_\varepsilon|k dy &= \int_{\mathbb{R}^d} (G_\varepsilon - bg_\varepsilon)\text{sign}g_\varepsilon k dy \\ &\leq \int_{\mathbb{R}^d} |G_\varepsilon|k dy - \int_{\mathbb{R}^d} |g_\varepsilon|bk dy. \end{aligned} \tag{3.13}$$

Here, $\text{sign}g_\varepsilon = 1$ if $g_\varepsilon > 0$, $\text{sign}g_\varepsilon = 0$ if $g_\varepsilon = 0$, $\text{sign}g_\varepsilon = -1$ if $g_\varepsilon < 0$. Finally, we can build (by a standard truncation and regularization by convolution process) the sequences (G_ε) and $g_\varepsilon(0)$ such that furthermore $G_\varepsilon \rightarrow G, g_\varepsilon(0) \rightarrow g(0)$ in the weak sense of measures in $M_k^1, \|G_\varepsilon(s)\|_{M_k^1} \leq \|G(s)\|_{M_k^1}$ for a.e. $s \in (0, T), \|g_\varepsilon(0)\|_{M_k^1} \leq \|g(0)\|_{M_k^1}$. By the previous uniqueness argument we have $g_\varepsilon \rightarrow g$ in the weak sense of measure and we get (3.12) by passing to the limit in (3.13). This ends the proof of Step 1.

Step 2: End of the proof of Lemma 3.2.

Consider two even solutions $f_1, f_2 \in C([0, T]; M_k^1(\mathbb{R}^d)) \cap L^\infty(0, T; M_{k_2}^1(\mathbb{R}^d))$ and let us denote $D = f_2 - f_1, S = f_1 + f_2$. By a standard algebraic computation D satisfies the following equation

$$\begin{aligned} \partial_t D &= \hat{Q}(f_2, f_2) - \hat{Q}(f_1, f_1) = \hat{Q}(D, S) \\ &= \hat{Q}_1(D, S) - SL(D) - L(S)D, \end{aligned}$$

where

$$\hat{Q}_i(\varphi, \psi) = \frac{1}{2}(Q_i(\varphi, \psi) - Q_i(\psi, \varphi)), \quad L(\varphi) := \int_{\mathbb{R}^d} a(p, p')\varphi(p') dp'.$$

Because of the assumption made on a and f we have $D \in C([0, T]; M_k^1 - weak), G := \hat{Q}_1(D, S) - SL(D) \in L^\infty(0, T; M_k^1)$ and $0 \leq b := L(S) \in C([0, T] \times \mathbb{R}^d)$ so that the first step implies

$$\begin{aligned} \frac{d}{dt} \|D\|_{M_k^1} &\leq \|(\hat{Q}_1(D, S) - SL(D))k\|_{M_k^1} - \|DkL(S)\|_{M_k^1} \\ &\leq \frac{1}{2} \iint a[k'' + k']|D(dp)|S(dp') - \frac{1}{2} \iint ak|D(dp)|S(dp') \\ &\leq \frac{1}{4} \iint \tilde{A}|D(dp)|S(dp') \leq \frac{C}{4} \|S\|_{M_{k_2}^1} \|D\|_{M_k^1}. \end{aligned}$$

Uniqueness follows by using the Gronwall lemma. □

Lemma 3.4 Consider a continuous aggregation rate $a : \mathbb{R}^{2d} \rightarrow \mathbb{R}_+$ which satisfies (3.8) (resp. (3.11)), (3.9) and such that, for some positive constant C :

$$a(p, p') \leq C(k(p) + k(p')) \quad \forall p, p' \in \mathbb{R}^d, \tag{3.14}$$

for the weight function $k(p) = 1 + |p|^2$. Then, for any given even (resp. radially symmetric) initial datum $f_{in} \in M_{2\alpha}^1(\mathbb{R}^d)$ there exists at least one even (resp. radially symmetric) solution $f \in C([0, T]; M^1(\mathbb{R}^d) - weak) \cap L^\infty(0, T; M_{2\alpha}^1(\mathbb{R}^d))$ to (3.1)–(3.4). This solution also satisfies that the map $t \mapsto M_\beta(t)$ is decreasing for any $\beta \in [0, 1]$.

Remark 3.5 It is likely that by adapting some arguments introduced in [21], see also [8, 17], for any even (resp. radially symmetric) initial datum $f_{in} \in L_{2\alpha}^1(\mathbb{R}^d)$ the approximating solution $f_n(t, \cdot)$ built in the proof below is a Cauchy sequence in $C([0, T; L^1(\mathbb{R}^d))$ so that we may conclude $f \in C([0, T]; L^1(\mathbb{R}^d)) \cap L^\infty(0, T; L_{2\alpha}^1(\mathbb{R}^d))$.

Proof of Lemma 3.4 We define the sequence of bounded aggregation rates $a_n := a \wedge n$, for which classically fixed point argument (see for instance [13] which deals with some similar situation) implies the existence of a unique even (resp. radially symmetric) solution $f_n \in C([0, T]; L_{2\alpha}^1(\mathbb{R}^d))$ to (3.1)–(3.4) associated with a_n for any initial datum $f_{in,n} \in L_{2\alpha+2}^1(\mathbb{R}^d)$, $\alpha \in \mathbb{N}$, $\alpha \geq 2$. Then, we have for any $\beta \in \mathbb{N}^*$, $\beta \leq \alpha$

$$\begin{aligned} & \frac{d}{dt} \int f_n(1 + |p|^{2\beta}) \\ &= \frac{1}{2} \int f_n f'_n a_n [(|p|^2 + 2p \cdot p' + |p'|^2)^\beta - |p|^{2\beta} - |p'|^{2\beta} - 1] \\ &= \int f_n f'_n a_n [2\beta p \cdot p' |p|^{2(\beta-1)} - 1/2] \\ & \quad + \sum \mu_{\beta_1, \beta_2, \beta_3} \int f_n f'_n a_n (p \cdot p')^{\beta_1} |p|^{2\beta_2} |p'|^{2\beta_3}, \end{aligned}$$

where in the last sum the integers $\beta_1, \beta_2, \beta_3$ are such that $\beta_1 + \beta_2 + \beta_3 = \beta$ and must satisfy also: either $\beta_1 \geq 2$, or $\beta_2 \geq 1$ and $\beta_3 \geq 1$. This implies: $|p \cdot p'|^{\beta_1} |p|^{2\beta_2} |p'|^{2\beta_3} \leq |p|^{2\beta'} |p'|^{2(\beta-\beta')}$ with $1 \leq \beta' \leq \beta - 1$. Since we also have

$$\begin{aligned} & \int f_n f'_n a_n p \cdot p' |p|^{2(\beta-1)} \\ &= \int_{p \cdot p' > 0} f_n f'_n (a(p, p') \wedge n - a(-p, p') \wedge n) p \cdot p' |p|^{2(\beta-1)} \leq 0, \end{aligned}$$

we conclude with

$$\frac{d}{dt} \int f_n(1 + |p|^{2\beta}) \leq \sum_{1 \leq \beta' \leq \beta-1} \mu_{\beta'} \int f_n f'_n a |p|^{2\beta'} |p'|^{2(\beta-\beta')}. \tag{3.15}$$

When $\beta = 1$ the set of admissible values of β' is empty, and we recover a result from [9]

$$\frac{d}{dt} \int f_n(1 + |p|^2) \leq 0,$$

so that

$$\sup_{[0, T]} \|f_n\|_{L_k^1} \leq \|f_{in,n}\|_{L_k^1}. \tag{3.16}$$

When $\beta \geq 2$, gathering (3.14), (3.15) and (3.16), we easily conclude by a iterative argument that

$$\sup_{[0, T]} \|f_n\|_{L^1_{k^\beta}} \leq C_T(\beta, \|f_{in, n}\|_{L^1_{k^\beta}}). \tag{3.17}$$

Considering a sequence $(f_{in, n})$ such that $f_{in, n} \rightharpoonup f_{in}$ in the weak sense of measure and $\|f_{in, n}\|_{L^1_{k^\beta}}$ remains bounded, we easily pass to the limit in the equation satisfied by f_n thanks to (3.17). The fact that $t \mapsto M_\beta(t)$ is decreasing follows from the fact that $p \mapsto |p|^\beta$ is a sub-additive function when $\beta \in [0, 1]$, so that $\Delta_\beta \leq 0$ and then $d/dt M_\beta(t) \leq 0$. \square

Proof of the existence and uniqueness part in Theorem 3.1 It is clear that $a(p, p') = |p - p'|^\gamma$ satisfies (3.8), (3.9), the first inequality in (3.10) and (3.14). Moreover, the second inequality in (3.10) holds since we have

$$\begin{aligned} \tilde{\Delta}_2(p, p') &= |p - p'|^\gamma (|p + p'|^2 + |p'|^2 - |p|^2 + 1) \\ &\quad + |p + p'|^\gamma (|p - p'|^2 + |p'|^2 - |p|^2 + 1) \\ &= 2(|p - p'|^\gamma - |p + p'|^\gamma)p \cdot p' \\ &\quad + (|p - p'|^\gamma + |p + p'|^\gamma)(2|p'|^2 + 1), \end{aligned}$$

where the first term is non-positive and the second term is bounded by say $8(k')^2k$, using that $|p \pm p'|^\gamma \leq 2(|p|^\gamma + |p'|^\gamma)$. We conclude by using Lemma 3.2 and Lemma 3.4. \square

3.2 Proof of the Rate Decay Part in Theorem 3.1 when $\gamma < 2$

For an even initial datum $f_{in} \in M^1_4(\mathbb{R}^d)$ we consider the unique even solution $f \in C([0, T]; M^1 - weak) \cap L^\infty(0, T; M^1_4), \forall T$, given by Theorem 3.1(i). It satisfies the moment equation

$$\frac{d}{dt} M_\gamma = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f f' \Delta_\gamma dp dp' = \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f f' \tilde{\Delta}_\gamma dp dp', \tag{3.18}$$

with

$$-\Delta_\gamma(p, p') = |p - p'|^\gamma [|p + p'|^\gamma - |p|^\gamma - |p'|^\gamma]$$

and

$$\begin{aligned} -\tilde{\Delta}_\gamma(p, p') &= |p - p'|^\gamma [|p + p'|^\gamma - |p|^\gamma - |p'|^\gamma] \\ &\quad + |p + p'|^\gamma [|p - p'|^\gamma - |p|^\gamma - |p'|^\gamma]. \end{aligned} \tag{3.19}$$

We split the proof of Theorem 3.1(ii) in several steps.

Step 1. On the one hand, for any given $A > 0$ and any $p, p' \in \mathbb{R}^d$ such that $A^{-1}|p'| \leq |p| \leq A|p|$ we easily get

$$\begin{aligned} |\Delta_\gamma(p, p')| &\leq (|p| + |p'|)^\gamma \max[(|p| + |p'|)^\gamma, |p|^\gamma + |p'|^\gamma] \\ &\leq 2^4 \max(|p|, |p'|)^{2\gamma} \leq 2^4 A^\gamma (|p||p'|)^\gamma. \end{aligned} \tag{3.20}$$

On the other hand, we define $M := \max(|p|, |p'|)$, $m := \min(|p|, |p'|)$, $x := m/M \in [0, 1]$, $\varepsilon := \hat{p} \cdot \hat{p}' \in [-1, 1]$ and we compute (in the first line we have assumed that $|p| = M$ which is not a restriction to the generality because of the symmetry of $\tilde{\Delta}_\gamma$)

$$\begin{aligned}
 -\tilde{\Delta}_\gamma(p, p') &= M^{2\gamma} \{ |\hat{p} - x\hat{p}'|^\gamma [1 + x^\gamma - |\hat{p} + x\hat{p}'|^\gamma] \\
 &\quad + |\hat{p} + x\hat{p}'|^\gamma [1 + x^\gamma - |\hat{p} - x\hat{p}'|^\gamma] \} \\
 &= M^{2\gamma} \{ (1 + x^\gamma)[(1 + 2\varepsilon x + x^2)^{\gamma/2} + (1 - 2\varepsilon x + x^2)^{\gamma/2}] \\
 &\quad - 2(1 + 2\varepsilon x + x^2)^{\gamma/2}(1 - 2\varepsilon x + x^2)^{\gamma/2} \} \\
 &= M^{2\gamma} \{ 2x^\gamma + \mathcal{O}(x^2) \} \leq 3M^{2\gamma} x^\gamma = 3(|p||p'|)^\gamma
 \end{aligned} \tag{3.21}$$

uniformly on $\varepsilon \in [-1, 1]$ and $x \leq A_0^{-1}$ for $A_0 \geq 1$ large enough.

Using (3.20) and (3.21) we obtain

$$\frac{1}{4} \tilde{\Delta}_\gamma(p, p') \geq -k_1 |p|^\gamma |p'|^\gamma \quad \forall p, p' \in \mathbb{R}^d,$$

with $k_1 := \max(3/4, 2^3 A_0^\gamma)/4$, and (3.18) then implies

$$\frac{d}{dt} M_\gamma \geq -k_1 M_\gamma^2.$$

The first inequality in (3.6) follows straightforwardly by integrating this differential equation.

Step 2. We still use the variables M, x, ε introduced in Step 1. We also define $r > 0$ and $u \in [0, 1]$ by setting $r^2 := |p|^2 + |p'|^2$ and $u := 2p \cdot p'/r^2$, so that $|p \pm p'|^2 = r^2(1 \pm u)$. Splitting the positive and the negative terms in identity (3.19), we have

$$\begin{aligned}
 -\tilde{\Delta}_\gamma(p, p') &= (|p|^\gamma + |p'|^\gamma)(|p - p'|^\gamma + |p + p'|^\gamma) - 2|p - p'|^\gamma |p + p'|^\gamma \\
 &= r^{2\gamma} \left\{ \frac{(|p|^2)^{\gamma/2} + (|p'|^2)^{\gamma/2}}{(|p|^2 + |p'|^2)^{\gamma/2}} [(1 + u)^{\gamma/2} + (1 - u)^{\gamma/2}] \right. \\
 &\quad \left. - 2(1 + u)^{\gamma/2}(1 - u)^{\gamma/2} \right\}.
 \end{aligned}$$

Since $\gamma/2 \in [0, 1]$, the map $x \mapsto x^{\gamma/2}$ is sub-additive, and we obtain

$$\begin{aligned}
 -\tilde{\Delta}_\gamma(p, p') &\geq r^{2\gamma} \{ [(1 + u)^{\gamma/2} + (1 - u)^{\gamma/2}] - 2(1 + u)^{\gamma/2}(1 - u)^{\gamma/2} \} \\
 &\geq M^{2\gamma} (1 + u)^{\gamma/2} (1 - u)^{\gamma/2} \phi(u), \\
 \phi(u) &:= [(1 - u)^{-\gamma/2} + (1 + u)^{-\gamma/2}] - 2.
 \end{aligned}$$

We easily verify that ϕ is increasing on $[0, 1]$ so that $\phi(u) > \phi(0) = 0$ for any $u \in [-1, 1]$, $u \neq 0$. Coming back to the variables M, x and ε , that is $\phi(u) > 0$ for any $p, p' \in \mathbb{R}^d$ such that the associated variables M, x and ε satisfy $M > 0, x > 0$ and $\varepsilon \neq 0$. Moreover, when $\varepsilon = 0$ (p and p' are orthogonal vectors) we also have

$$\begin{aligned}
 -\tilde{\Delta}_\gamma(p, p') &= 2(|p|^2 + |p'|^2)^{\gamma/2} [|p|^\gamma + |p'|^\gamma - (|p|^2 + |p'|^2)^{\gamma/2}] \\
 &\geq 2M^{2\gamma} [1 + x^\gamma - (1 + x^2)^{\gamma/2}] > 0
 \end{aligned}$$

for any $p, p' \in \mathbb{R}^d$ such that the associated variables M and x satisfy $M > 0, x > 0$, because the function $z \mapsto z^{\gamma/2}$ is strictly sub-additive on \mathbb{R}_+ , that is $(z + z')^{\gamma/2} < z^{\gamma/2} + (z')^{\gamma/2}$ for any $z, z' > 0$. From these two lower bounds on $-\tilde{\Delta}_\gamma$, we obtain

$$-\tilde{\Delta}_\gamma(p, p') \geq M^{2\gamma} \psi(x, \varepsilon) \tag{3.22}$$

with $\psi(x, \varepsilon) > 0$ for any $x > 0$ and $\varepsilon \in [-1, 1]$.

Next, coming back to (3.21), we also deduce

$$-\tilde{\Delta}_\gamma(p, p') = M^{2\gamma} \{2x^\gamma + \mathcal{O}(x^2)\} \geq M^{2\gamma} x^\gamma \tag{3.23}$$

uniformly on $\varepsilon \in [-1, 1]$ and $x \leq A_0^{-1}$ for $A_0 \geq 1$ large enough. We deduce from (3.22) and (3.23) that for some constant $k_2 > 0$ we have

$$\forall p, p' \in \mathbb{R}^d \quad -\frac{1}{4} \tilde{\Delta}_\gamma \geq k_2 M^{2\gamma} x^\gamma = k_2 (|p||p'|)^\gamma,$$

and (3.18) then implies

$$\frac{d}{dt} M_\gamma \leq -k_2 M_\gamma^2.$$

The second inequality in (3.6) is again obtained by integrating this differential equation.

3.3 The Case $a(y, y') = |p - p'|, d = 1$

In the particular case $d = 1$ and $\gamma = 1$, it is possible to estimate more precisely the decay rate of the first moment M_1 . It is also possible to estimate the decay rates of several other moments.

Lemma 3.6 *Assume $a(y, y') = |p - p'|$ and $d = 1$. For any even initial data $f_{in} \in M_3^1(\mathbb{R})$ the unique solution $f \in C([0, T]; M^1(\mathbb{R})) \cap L^\infty(0, T; M_3^1(\mathbb{R}))$ of (3.1)–(3.4) given by Theorem 3.1 satisfies for any $t \geq 0$*

$$\begin{aligned} & \max \left(\frac{M_0(0)}{(1 + M_1(0)t/2)^2}, \frac{2^{3/2} M_0(0)}{(2 + 3M_3^{1/3}(0)t)^{3/2}} \right) \\ & \leq M_0(t) \leq \frac{M_0(0)}{(1 + M_1(0)t)^{1/2}}, \end{aligned} \tag{3.24}$$

$$\frac{1}{M_1(0)^{-1} + t} \leq M_1(t) \leq \frac{1}{M_1(0)^{-1} + t/2}, \tag{3.25}$$

$$\frac{M_2(0)}{(1 + M_1(0)t/2)^2} \leq M_2(t) \leq M_2(0), \tag{3.26}$$

$$\frac{M_3(0)}{(1 + M_1(0)t/2)^2} \leq M_3(t) \leq M_3(0). \tag{3.27}$$

Remark 3.7 The estimate (3.25) on $M_1(t)$ gives the exact value of the power of t at which the first moment decays for t large. That is not the case for the estimates on $M_\alpha, \alpha = 0, 2, 3$ which are actually rather partial. They do not even allow to obtain the limit of any of the quotients of moments $M_\alpha(t)/M_1(t)$ for $\alpha = 0, 2, 3$ as $t \rightarrow \infty$. The value of such limits

would indicate whether the solution $f(t)$ has a tendency to concentrate or to spread as t increases (see also below the discussion concerning the case $\gamma = 2$).

Remark 3.8 The approximation of the relative velocity $|v - v'|$ in the collision kernel $a(y, y')$ by the root mean squared velocity $V(t)$ has been sometimes considered. The authors of [29] have treated such an approximation in the simplified equation where the collision kernel $a(y, y')$ is exactly $a = |v - v'|$. They have proved that the corresponding particle density decays like t^{-2} , that is exactly the power predicted by the mean-field no-correlation approach. A different approximation, called “very hard potential” has also been considered by these authors. In particular they remark that the numerical value obtained for the decay rate of the particle density in the simplified model is between the two values predicted by the mean-field no-correlation approach in the hard and very hard potential approximation.

Inspired by this one could be tempted to approximate in (3.1)–(3.5) with $\gamma = 1$ the collision kernel $|p - p'|$ by the root mean squared momentum $P(t)\sqrt{m_2(t)}$ where m_2 denotes the second moment of the solution of that modified equation. We easily compute:

$$m_2(t) \equiv \left(m_2(0)^{1/2} + \frac{m_1^2 t}{2}\right)^2 \quad \text{and} \quad m_0(t) = \frac{1}{\frac{1}{m_0(0)} + \frac{m_2^{1/2}(0)t}{2} + \frac{m_1^2 t^2}{8}}.$$

It is worth emphasizing that $M_0(t)$ (the number of particles at time t for the equation with rate $a(p, p') = |p - p'|$) and $m_0(t)$ (the number of particles at time t for the equation with rate $a(p, p') = P(t)$) have definitely not the same long time behavior. As a conclusion, such an approximation is not a good one here.

Proof of Lemma 3.6 Let us denote $M = \max(|p|, |p'|)$ and $m = \min(|p|, |p'|)$. We systematically exploit the differential equation

$$\frac{d}{dt} M_\alpha = \frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} f f' \Delta_\alpha dp dp' \tag{3.28}$$

with

$$\Delta_\alpha := [M - m][(M + m)^\alpha - M^\alpha - m^\alpha] + [M + m][(M - m)^\alpha - M^\alpha - m^\alpha].$$

Step 1: We suppose $\alpha = 1$. In that case

$$\Delta_1 = -2(M + m)m,$$

from where we deduce

$$\begin{aligned} \frac{d}{dt} M_1(t) &= -\frac{M_1^2(t)}{2} - \frac{B_1(t)}{2}, \\ B_1(t) &:= \int_{\mathbb{R}} \int_{\mathbb{R}} f f' \{\min(|p|, |p'|)\}^2 dp dp'. \end{aligned}$$

Since $0 \leq \{\min(|p|, |p'|)\}^2 \leq |p||p'|$, we have $0 \leq B_1(t) \leq M_1^2(t)$ and therefore

$$-M_1^2(t) \leq \frac{d}{dt} M_1(t) \leq -\frac{M_1^2(t)}{2}, \tag{3.29}$$

from where (3.25) follows.

Step 2: We suppose $\alpha = 0$. Then, since

$$\Delta_0 = -2M,$$

we have

$$\frac{d}{dt}M_0(t) = -\frac{B_0(t)}{2}, \quad B_0(t) := \int_{\mathbb{R}} \int_{\mathbb{R}} ff' \max(|p|, |p'|) dp dp'. \tag{3.30}$$

Using $|p| \leq \max(|p|, |p'|) \leq |p| + |p'|$, we deduce $M_0M_1 \leq B_0 \leq 2M_0M_1$ and then

$$-M_0M_1 \leq \frac{d}{dt}M_0 \leq -\frac{1}{2}M_0M_1. \tag{3.31}$$

By the previous estimate (3.25) on $M_1(t)$ we get

$$-\frac{M_0(t)}{M_1^{-1}(0) + t/2} \leq \frac{d}{dt}M_0(t) \leq -\frac{M_0(t)}{2(M_1^{-1}(0) + t)},$$

and we obtain the first lower estimate as well as the upper bound in (3.24).

Step 3: The case $\alpha = 2$. We deduce from

$$\Delta_2 = -4mM^2$$

that:

$$\frac{d}{dt}M_2(t) = -B_2(t), \quad B_2(t) := \int_{\mathbb{R}} \int_{\mathbb{R}} ff' \min(|p|, |p'|)|p||p'| dp dp'.$$

Using that $0 \leq \min(|p|, |p'|)|p||p'| \leq |p|^2|p'|$ together with (3.25), we obtain

$$-M_2 \frac{1}{M_1(0)^{-1} + t/2} \leq -M_2M_1 \leq \frac{d}{dt}M_2(t) \leq 0,$$

and (3.26) follows.

Step 4: The case $\alpha = 3$. From the following estimates on Δ_3

$$0 \geq \Delta_3 = -2Mm^3 - 2m^4 \geq -4Mm^3 \geq -4|p|^3|p'|,$$

we deduce

$$0 \geq \frac{d}{dt}M_3(t) \geq -M_1M_3,$$

which again implies (3.27).

Step 5: Suppose $\alpha = 0$ again. We complete now the lower estimate of M_0 in (3.24). To this end we write for any $\varepsilon > 0$

$$\begin{aligned} \frac{d}{dt}M_0 &= -\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} ff'|p' - p| dp dp' \\ &\geq -\frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} ff' \left(\varepsilon + \frac{1}{\varepsilon}|p - p'|^2 \right) dp dp' \\ &\geq -\frac{\varepsilon}{4}M_0^2 - \frac{2}{\varepsilon}M_0M_2. \end{aligned}$$

By interpolation we have $M_2(t) \leq M_0^{1/3}(t)M_3^{2/3}(t)$. Since, by (3.27), $M_3(t) \leq M_3(0)$ for all $t > 0$ we deduce $M_2(t) \leq M_0^{1/3}(t)M_3^{2/3}(0)$. Therefore

$$\frac{d}{dt}M_0(t) \geq -\frac{\varepsilon}{4}M_0^2 - \frac{2}{\varepsilon}M_0^{4/3}M_3^{2/3}(0).$$

We now chose $\varepsilon \equiv \varepsilon(t) > 0$ such that $\varepsilon M_0^2 = \frac{1}{\varepsilon}M_0^{4/3}M_3^{2/3}(0)$, or equivalently $\varepsilon = M_0^{-1/3}M_3^{1/3}(0)$. With that choice of $\varepsilon(t)$ the equation reads

$$\frac{d}{dt}M_0(t) \geq -\frac{9}{4}M_3^{1/3}(0)M_0^{5/3},$$

and the second lower estimate in (3.24) follows. □

Remark 3.9 In the last step, we may also argue as follows. Combining the estimate $\max(|p|, |p'|) \geq (|p||p'|)^{1/2}$, (3.27), the differential equation (3.30) and the interpolation estimate $M_1^{5/2} \leq M_{1/2}^2M_3^{1/2}$ we obtain:

$$\frac{d}{dt}M_0 \leq -\frac{1}{dt}M_3^{1/2}(0)M_1^{5/2}(t).$$

Using (3.25) we recover the second lower estimate in (3.24).

3.4 The Case $a = |p - p'|^2$

When $\gamma = 2$ and $d \in \mathbb{N}^*$, the family of moment equations may be closed, for all the “even” moments $M_{2\alpha}$, $\alpha \in \mathbb{N}$. This allows to prove a non-existence result of self similar solutions in that case. We first obtain in the next lemma the exact expressions of the even moments of order less that or equal to four of the solutions to (3.1)–(3.4).

Lemma 3.10 *Assume $a(y, y') = |p - p'|^2$ and $d \in \mathbb{N}^*$. Then, there exist numerical constants $k_d \in (0, \infty)$, $k_1 := 2$, such that for any radially symmetric initial datum $f_{in} \in M_6^1(\mathbb{R}^d)$ the unique radially symmetric solution $f \in C([0, T]; M^1(\mathbb{R}^d)) \cap L^\infty(0, T; M_6^1(\mathbb{R}^d))$ of (3.1)–(3.4) given by Theorem 3.1 satisfies for any $t \geq 0$*

$$M_0(t) = \frac{M_0(0)}{(M_2(0)^{-1} + 2k_d t)^{1/(2k_d)}}, \tag{3.32}$$

$$M_2(t) = \frac{1}{M_2(0)^{-1} + 2k_d t}, \tag{3.33}$$

$$M_4(t) = M_4(0)(M_2(0)^{-1} + 2k_d t)^{\frac{1}{k_d} - 2}. \tag{3.34}$$

Proof of Lemma 3.10 We proceed in several steps.

Step 1: If $\alpha = 2$. Using the fact that f is radially symmetric (so that the odd moments of f vanish) and the notations $p = r\sigma$, $r = |p|$, $p' = r'\sigma'$, $r' = |p'|$, the fundamental moment identity (3.7) implies

$$\begin{aligned}
 \frac{d}{dt} M_2 &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f f' [|p|^2 - 2p \cdot p' + |p'|^2] (2p \cdot p') dp dp' \\
 &= -2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f f' [p \cdot p']^2 dp dp' \\
 &= -2 \int_0^\infty \int_0^\infty f(r) f(r') r^{d+1} (r')^{d+1} dr dr' \\
 &\quad \times \int_{S^{d-1}} \int_{S^{d-1}} [\sigma \cdot \sigma']^2 d\sigma d\sigma' \\
 &= -2k_d M_2^2,
 \end{aligned}$$

with

$$\begin{aligned}
 k_d &:= \omega_d^{-2} \left(\int_{S^{d-1}} \int_{S^{d-1}} [\sigma \cdot \sigma']^2 d\sigma d\sigma' \right) \\
 &= \omega_d^{-1} \int_{S^{d-1}} \sigma_1^2 d\sigma.
 \end{aligned}$$

We compute $k_1 = 1$, $k_2 = 1/2$. The expression (3.33) immediately follows by integrating that ODE.

Step 2: If $\alpha = 0$. In that case the fundamental moment identity (3.7) and the fact that f is radially symmetric imply

$$\begin{aligned}
 \frac{d}{dt} M_0 &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f f' [|p|^2 - 2p \cdot p' + |p'|^2] (-1) dp dp' \\
 &= -M_2 M_0.
 \end{aligned}$$

Integrating that ODE with the help of (3.33) we get (3.32).

Step 3: If $\alpha = 4$. When $\alpha = 4$, the fundamental moment identity (3.7) and the fact that f is radially symmetric imply

$$\begin{aligned}
 \frac{d}{dt} M_4 &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f f' [|p|^2 - 2p \cdot p' + |p'|^2] \\
 &\quad \times [4(p \cdot p')^2 + 8|p|^2(p \cdot p') + 2|p|^2|p'|^2] dp dp' \\
 &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f f' \{ [2|p|^2] [4(p \cdot p')^2 + 2|p|^2|p'|^2] - 16|p|^2(p \cdot p')^2 \} dp dp' \\
 &= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f f' \{ |p|^4 |p'|^2 - 2|p|^2(p \cdot p')^2 \} dp dp' \\
 &= (2 - 4k_d) M_2 M_4.
 \end{aligned}$$

Integrating that ODE with the help of (3.33) we get (3.34). □

Remark 3.11 It is straightforward to check that if $\Phi(t, m, p)$ solves (1.1), (1.3), (1.4) with $a(y, y') \equiv |p - p'|^\gamma$, so does

$$\Phi_\rho(t, m, p) = \rho^{-\lambda} f(\rho t, \rho^{-\delta} m, \rho^{-\nu} p) \tag{3.35}$$

for all $\rho > 0$, whenever the exponents satisfy:

$$\lambda = \nu(\gamma + 1) + \delta - 1. \tag{3.36}$$

This may suggest the existence of self similar solutions of (1.1), (1.3), (1.4) with $a(y, y') \equiv |p - p'|^\nu$ of the form :

$$\Phi(t, m, p) = t^\lambda \Theta(t^\delta m, t^\nu p) \tag{3.37}$$

for some function Θ . Therefore the function

$$\begin{aligned} g(t, p) &= \int_{\mathbb{R}^d} t^\lambda \Theta(t^\delta m, t^\nu p) dm = t^{\lambda-\delta} \int_{\mathbb{R}^d} \Theta(m, t^\nu p) dm \\ &=: t^\mu G(t^\nu p) \end{aligned} \tag{3.38}$$

with $\mu = \lambda - \delta = \nu(\gamma + 1) - 1$ would be a self similar solution of (3.1)–(3.5) and its moments of order $\alpha \in \mathbb{R}$, would then be:

$$M_\alpha(g(t, \cdot)) = M_\alpha(G) t^{\mu - (d+\alpha)\nu}. \tag{3.39}$$

On the other hand, by Lemma 3.10, when $\gamma = 2$ the solutions f of (3.1)–(3.4) with initial data $f_{in} \in M_6^1(\mathbb{R})$ satisfy

$$M_0(f(t, \cdot)) \sim C'_0 t^{-\frac{1}{2k_d}}, \quad M_2(f(t, \cdot)) \sim C'_2 t^{-1}, \quad M_4(f(t, \cdot)) \sim C'_4 t^{\frac{1}{k_d}-2}. \tag{3.40}$$

as $t \rightarrow +\infty$ for some positive constants C'_0, C'_2 and C'_4 . It is easy to check that to have both (3.39) and (3.40) requires :

$$(3 - d)\nu - 1 = -\frac{1}{k_d}; \quad (1 - d)\nu - 1 = -1; \quad -(1 + d)\nu - 1 = \frac{1}{k_d} - 2$$

that is impossible. We deduce that when $\gamma = 2$, (3.1)–(3.5) has no self similar solution of the form (3.38) with self-similar profile $G \in M_6^1(\mathbb{R})$. Therefore (1.1), (1.3), (1.4) with $a(y, y') \equiv |p - p'|^\nu$ has no self similar solution of the form (3.37) with Θ such that $M_{1,0}(\Theta) < \infty$ and $M_{0,6}(\Theta) < \infty$.

Notice that in order for the function $\Theta(q)$ to satisfy the last condition $M_{0,6}(\Theta) < \infty$ it must, in particular, decay fast enough to zero as $|q| \rightarrow \infty$. Therefore, Lemma 3.10 does not exclude the existence of self similar solutions of the form (3.38) with “fat tailed” self-similar profiles Θ such that $M_{0,k}(\Theta) = \infty$ for $k \geq 6$.

Remark 3.12 Consider again any solution f of (3.1)–(3.4) with initial data $f_{in} \in M_6^1$ and suppose, only for the sake of simplicity, that we are in the case $d = 1$. Then the moments $M_\gamma(f(t))$ for $0 \leq \gamma \leq 2\alpha$ satisfy:

$$\frac{d}{dt} M_{2\alpha} = \sum_{\beta=1}^{\alpha-1} \binom{2\alpha}{2\beta} M_{2\beta} M_{2(\alpha+1-\beta)} - \sum_{\beta=0}^{\alpha-1} \binom{2\alpha}{2\beta+1} M_{2\beta+2} M_{2(\alpha-\beta)}.$$

In particular:

$$\frac{d}{dt} M_6 = 3M_2M_6 - 5M_4^2.$$

When $M_2(0) = 1/2$ (only for the sake of simplicity again), the solution is

$$M_6(t) = \left(M_6(0) - 2M_4(0)^2 + \frac{2M_4(0)^2}{(1+t)^{5/4}} \right) (1+t)^{3/2} \quad \forall t \geq 0,$$

with, by Holder’s inequality, $M_6(0) - 2M_4(0)^2 > 0$. It then follows that the moments of f satisfy the following

$$M_0(t) \sim \kappa_0 t^{-1/2}, \quad \frac{M_2(t)}{M_0(t)} \sim \kappa_1 t^{-1/2}, \quad \frac{M_6(t)}{M_0(t)} \sim \kappa_2 t^{3/2}, \tag{3.41}$$

as $t \rightarrow +\infty$, for some positive constants $\kappa_i, i = 0, 1, 2$. We notice that the “mean second moment” tends to 0 as $t \rightarrow +\infty$. The behavior of the mean second moment M_2/M_0 is then comparable with the behavior of the energy (second moment) of the solutions to the inelastic Boltzmann equation that decreases as $t \rightarrow +\infty$. The opposite will be true for a model considered in Sect. 4. On the other hand, (3.41) shows that the behavior of the mean sixth moment M_6/M_0 is similar to that of the high moments of the solutions of the Smoluchoski equation that increase as $t \rightarrow +\infty$.

4 The Mass Dependence Case $a = a(m, m')$

Consider now the problem (1.1)–(1.4) where the kernel $a(y, y')$ only depends on the masses of the particles, namely

$$a(y, y') = a(m, m'), \tag{4.1}$$

and introduce the associated Smoluchowski equation

$$\begin{aligned} \frac{\partial F}{\partial t}(t, m) &= \frac{1}{2} \int_0^m F(t, m - m') F(t, m') a(m - m', m') dm' \\ &\quad - \int_0^\infty F(t, m) F(t, m') a(m, m') dm'. \end{aligned} \tag{4.2}$$

For any function $\psi \in L^1(\mathbb{R}^3)$ we define the Fourier transform \mathcal{F} and the inverse Fourier transform \mathcal{F}^{-1} by

$$\begin{aligned} \hat{\psi}(\eta) &= (\mathcal{F}\psi)(\eta) = \int_{\mathbb{R}^3} \psi(p) e^{-ip \cdot \eta} dp, \\ (\mathcal{F}^{-1}\psi)(p) &= (2\pi)^{-3} \int_{\mathbb{R}^3} \psi(\eta) e^{ip \cdot \eta} dp. \end{aligned}$$

Theorem 4.1 *For any continuous function a on \mathbb{R}^3 , homogeneous of degree θ^{-1} , $\theta \in (0, \infty)$, and such that $\varphi := \mathcal{F}^{-1}(e^{-a(\cdot)}) \geq 0$, and for any solution $F \equiv F(t, m)$ to the coagulation equation (4.2) with coagulation kernel $a(m, m')$, the function $f(t, m, p)$ defined by*

$$f(t, m, p) = m^{-3\theta} F(t, m) \varphi\left(\frac{p}{m^\theta}\right), \tag{4.3}$$

is a solution of (1.1), (1.3), (1.4) for the same aggregation kernel.

Remark 4.2 The functions defined on \mathbb{R}^d that are the Fourier transform of non nonnegative measures are, by a theorem of S. Bochner (see for example [25]), the so called positive definite functions. Examples of such functions are $e^{-\beta|p|^r}$ for all $\beta > 0$ and $r \in (0, 2]$. We deduce that $a(p) = |p|^{1/\theta}$, with $\theta \geq 1/2$ are admissible examples in Theorem 4.1. For $\theta = 1/2$ the velocity distribution function is then of Maxwellian type. In the case $\theta = 1$ the velocity distribution function is of the type Lynden-Bell obtained in [19]. Another example is $a(p) = |p_1| + |p_2| + |p_3|$ for which $\theta = 1$.

Remark 4.3 Notice that the initial data of the solutions (4.3) in Theorem 4.1 are of the form $f(0, m, p) = m^{-3\theta} F_{in}(m)\varphi(p/m^\theta)$ for some F_{in} . Theorem 4.1 is not therefore a general existence result of solutions to the Cauchy problem associated to (1.1), (1.3), (1.4).

Proof of Theorem 4.1 We have to check that the function $f(t, m, p)$ defined by (4.3) solves (1.1), (1.3), (1.4). We start with writing

$$\begin{aligned} \frac{\partial f}{\partial t} &= m^{-3\theta} \varphi\left(\frac{p}{m^\theta}\right) \frac{\partial F}{\partial t} \\ &= m^{-3\theta} \varphi\left(\frac{p}{m^\theta}\right) \left[\frac{1}{2} \int_0^m F(t, m - m')F(t, m')a(m - m', m') dm' \right. \\ &\quad \left. - \int_0^\infty F(t, m)F(t, m')a(m, m') dm' \right]. \end{aligned} \tag{4.4}$$

On the one hand, using that

$$\int_{\mathbb{R}^3} \varphi(p) dp = \mathcal{F}(\varphi)(0) = e^{-a(0)} = 1,$$

the last term in (4.4) gives

$$\begin{aligned} &m^{-3\theta} \varphi\left(\frac{p}{m^\theta}\right) \int_0^\infty F(t, m)F(t, m')a(m, m') dm' \\ &= m^{-3\theta} \varphi\left(\frac{p}{m^\theta}\right) F(t, m) \int_0^\infty a(m, m')F(t, m') \int_{\mathbb{R}^3} (m')^{-3\theta} \varphi\left(\frac{p'}{m'^\theta}\right) dp' \\ &= f(t, m, p) \int_0^\infty \int_{\mathbb{R}^3} a(m, m')f(t, m', p') dp'. \end{aligned} \tag{4.5}$$

On the other hand, let us define the function

$$g(m, p) = m^{-3\theta} \varphi(p/m^\theta).$$

Using the definition of φ and the homogeneity of a , it satisfies for any $0 < m' < m$

$$\begin{aligned} \hat{g}(m, \eta) &= \hat{\varphi}(m^\theta \eta) = \exp(-a(m^\theta \eta)) = \exp(-ma(\eta)) \\ &= \exp(-m'a(\eta)) \exp(-(m - m')a(\eta)) \\ &= \hat{g}(m', \eta) \hat{g}(m - m', \eta), \end{aligned}$$

or coming back to the origin function

$$g(m, p) = \int_{\mathbb{R}^3} g(m', p')g(m - m', p - p') dp'.$$

Using that identity in the first (gain) term in (4.4), we get

$$\begin{aligned}
 & m^{-3\theta} \varphi\left(\frac{p}{m^\theta}\right) \int_0^m F(t, m - m') F(t, m') a(m - m', m') dm' \\
 &= g(m, p) \int_0^m F(t, m - m') F(t, m') a(m - m', m') dm' \\
 &= \int_{\mathbb{R}^3} \int_0^m F(t, m - m') g(m - m', p - p') F(t, m') g(m', p') a(m - m', m') dm' dp' \\
 &= \int_{\mathbb{R}^3} \int_0^m f(t, m - m', p - p') f(t, m', p') a(m - m', m') dm' dp'. \tag{4.6}
 \end{aligned}$$

We conclude that f satisfies (1.1), (1.3), (1.4) by gathering (4.4), (4.5) and (4.6). □

Remark 4.4 Solutions of the form $n(m, t)\varphi(m, v)$ or $n(t, m)e^{-m|v|^2}$ (which corresponds to $\theta = 1/2$) have been considered in previous references as for example [4, 5, 27]. Sometimes this form is obtained from physical arguments, sometimes it is postulated as a simplifying ansatz.

The previous Theorem is useful in order to prove the existence of self similar solutions for some kernels $a(m, m')$ as it is seen in the following corollary.

Corollary 4.5 *Suppose that a and θ are as in Theorem 4.1. Assume further that F is a self similar solution of the coagulation equation with coagulation kernel $a(m, m')$. Then the function f defined by (4.3) is a self similar solution of (1.1), (1.3), (1.4).*

Proof of Corollary 4.5 The hypothesis on F means that for some functions $\Phi, v(t)$ and $\mu(t)$ it may be written as:

$$F(t, m) = v(t)\Phi(\mu(t)m).$$

Therefore f is a self-similar function since it may be written as

$$\begin{aligned}
 f(t, m, p) &= m^{-3\theta} v(t)\Phi(\mu(t)m)\varphi\left(\frac{p}{m^\theta}\right) \\
 &= v(t)\mu(t)^{3\theta} (\mu(t)m)^{-3\theta} \Phi(\mu(t)m)\varphi\left(\frac{\mu(t)^\theta p}{(\mu(t)m)^\theta}\right) \\
 &= v(t)\mu(t)^{3\theta} \Psi(\mu(t)m, \mu(t)^\theta p)
 \end{aligned}$$

with $\Psi(M, P) = M^{-3\theta} \Phi(M)\varphi(P/M^\theta)$. □

Remark 4.6 The existence of self similar solutions of (1.1), (1.3), (1.4) corresponding to the case $\theta = 1/2$ of Corollary 4.5 had already been proved in [13].

Remark 4.7 Self similar solutions of the coagulation equation are well known to exist for the cases $a(m, m') = 1$, $a(m, m') = m + m'$ and $a(m, m') = mm'$. Their existence has been proved in [10] and [12], for several other kernels with homogeneity $\lambda < 1$. In that case they

are of the form:

$$F(t, m) = t^{-\frac{2}{1-\lambda}} \Phi\left(\frac{m}{t^{\frac{1}{1-\lambda}}}\right). \tag{4.7}$$

We deduce that, under the assumption of the Corollary 4.5, and for these kernels $a(m, m')$ with homogeneity $\lambda < 1$,

$$f(t, m, p) = t^{-\frac{2}{1-\lambda}} m^{-3\theta} \Phi\left(\frac{m}{t^{\frac{1}{1-\lambda}}}\right) \varphi\left(\frac{p}{m^\theta}\right) \tag{4.8}$$

is a self similar solutions to (1.1), (1.3), (1.4). A straightforward calculation yields

$$\begin{aligned} P_k(t) &= \int_{\mathbb{R}^d} \int_0^\infty |p|^k f(t, m, p) dm dp \\ &= t^{-\frac{1-k\theta}{1-\lambda}} \int_{\mathbb{R}^d} |P|^k \varphi(P) dP \int_0^\infty M^{k\theta} \Phi(M) dM. \end{aligned} \tag{4.9}$$

As a consequence, we have $P_0 \rightarrow 0, P_1 \rightarrow 0$ and more generally $P_k \rightarrow 0$ whenever $k < \theta^{-1}$ but $P_k/P_0 \rightarrow \infty$ for any $k > 0$ and $P_k \rightarrow \infty$ whenever $k > \theta^{-1}$. The interpretation in terms of the model is that the total number of particles in the gas and the total momentum of the gas decrease and tend to zero, while, for instance, the mean second moment P_2/P_0 tends to infinity as t tends to infinity. This behavior is quite similar to that of the solutions to the Smoluchowski equation (where the mean momentum moment $P_k/P_0 \rightarrow \infty$ for any $k > 0$) and is completely different to that discussed in Remark 3.11.

5 The Constant Case $a = 1$

We consider in this section the aggregation kernel $a = 1$. Equations (1.1), (1.3) (1.4) reads then:

$$\begin{aligned} \partial_t f(t, m, p) &= \frac{1}{2} \int_{\mathbb{R}^d} \int_0^m f(t, m - m', p - p') f(t, m', p') dm' dp' \\ &\quad - f(t, m, p) \int_{\mathbb{R}^d} \int_0^\infty f(t, m', p') dm' dp'. \end{aligned} \tag{5.1}$$

The first result is on the existence of self similar solutions.

Theorem 5.1 *Let $\Phi \in C^1(\mathbb{R}, \mathbb{R})$ such that:*

$$\lim_{\zeta \rightarrow 0, \xi \rightarrow 0} \zeta \Phi\left(\frac{|\xi|^2}{\zeta}\right) = 0 \tag{5.2}$$

and suppose that

$$g(y, x) = \mathcal{F}_\xi^{-1} \mathcal{L}_\zeta^{-1} \left(\frac{2}{2\zeta \Phi\left(\frac{|\xi|^2}{\zeta}\right) + 1} \right) \tag{5.3}$$

satisfies $g \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$. Then, for all real positive numbers $\beta_i, i = 1, \dots, d$ the function

$$t^{-\frac{d+4}{2}} g\left(\frac{m}{t}, \beta_1 \frac{p_1}{\sqrt{t}}, \dots, \beta_d \frac{p_d}{\sqrt{t}}\right) \tag{5.4}$$

is a self similar weak solution to (1.1), (1.3), (1.4) with $a = 1$. If moreover $g \in C^1(\mathbb{R}^+ \times \mathbb{R}^d)$ then it is a classical solution.

Remark 5.2 As it will be seen in Remark 5.3 below, it is easy to obtain different self similar solutions of (5.1) using Theorem 5.1. Notice nevertheless that these self similar solutions are not necessarily non negative.

Proof of Theorem 5.1 The formal argument leading to the expression (5.3) and condition (5.2) is the following. If we look for a self similar solution of (5.1) of the form (5.4) the function g must then solve:

$$\begin{aligned} -\frac{d+2}{2}g - y\partial_y g - \frac{1}{2}x \cdot \nabla_x g &= \frac{1}{2} \int_{\mathbb{R}^d} \int_0^y g(y-y', x-x')g(y', x') dy' dx' \\ &\quad - g \int_{\mathbb{R}^d} \int_0^\infty g(y', x') dy' dx'. \end{aligned} \tag{5.5}$$

We integrate this equation with respect to x and y and obtain

$$\int_{\mathbb{R}^d} \int_0^\infty g(y', x') dy' dx' = 2. \tag{5.6}$$

We now Fourier transform with respect to x and Laplace transform with respect to y :

$$\zeta \partial_\zeta \widehat{g} + \frac{1}{2}\xi \cdot \nabla_\xi \widehat{g} = \frac{1}{2}\widehat{g}^2 - \widehat{g}. \tag{5.7}$$

We divide by \widehat{g}^2 and define $G = 1/\widehat{g}$:

$$\zeta \partial_\zeta G + \frac{1}{2}\xi \cdot \nabla_\xi G = G - \frac{1}{2}. \tag{5.8}$$

The function G may then be any function of the form:

$$G(\zeta, \xi) = \zeta \Phi\left(\frac{|\xi|^2}{\zeta}\right) + \frac{1}{2} \tag{5.9}$$

for any arbitrary, derivable function Φ . Therefore we should have:

$$\widehat{g}(\zeta, \xi) = \frac{2}{2\zeta \Phi\left(\frac{|\xi|^2}{\zeta}\right) + 1}, \tag{5.10}$$

with, due to (5.6):

$$\lim_{\zeta \rightarrow 0, \xi \rightarrow 0} \frac{2}{2\zeta \Phi\left(\frac{|\xi|^2}{\zeta}\right) + 1} = 2 \iff \lim_{\zeta \rightarrow 0, \xi \rightarrow 0} \zeta \Phi\left(\frac{|\xi|^2}{\zeta}\right) = 0.$$

It is then straightforward to check that, given any function $\Phi \in C^1(\mathbb{R}, \mathbb{R})$ satisfying (5.2) the function G defined by (5.9) is such that G^{-1} satisfies (5.7). Therefore, if (5.3) defines a function $g \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$, the function g satisfies (5.5) in the weak sense of distributions in $\mathbb{R}^+ \times \mathbb{R}^d$ and the function (5.4) is a weak solution of (5.1). \square

Remark 5.3 If $\Phi(z) = z + 1$,

$$\begin{aligned} \hat{g}(\zeta, \xi) &= \frac{2}{2\zeta\left(\frac{|\xi|^2}{\zeta} + 1\right) + 1} = \frac{2}{2(|\xi|^2 + \zeta) + 1} \\ &= \mathcal{L}\left(\mathcal{F}\left(\frac{e^{-\frac{y}{2}}e^{-\frac{|x|^2}{4y}}}{\sqrt{2}\sqrt{y}}\right)\right) \end{aligned}$$

and then,

$$f(t, m, p) = t^{-(d+4)/2} \frac{1}{\sqrt{2}\sqrt{m}} e^{-\frac{m}{2t}} e^{-\frac{1}{4m} \sum_{i=1}^d \beta_i^2 p_i^2} \tag{5.11}$$

is a self similar solution of (5.1). This is, up to a constant, the profile of the self similar solution that appears in Theorem 5.4 below. It is possible to obtain other self similar solutions of (5.1). Some of them are explicit others are not. If, for example, $\Phi \equiv 1$ then $g(y, x) = e^{-y^2} \delta_{x=0}$. Another explicit example is for $\Phi(z) = z$ which gives $g(y, x) = \sqrt{\pi} \delta_{y=0} e^{-\frac{|x|^2}{2}}$. Notice that in all these three examples the solution g is non negative. On the other hand, if we take $\Phi(z) = \sqrt{z}$, the inverse Laplace transform, let us call it $h(y, \xi)$, is still explicit:

$$h(y, \xi) = \mathcal{L}_\zeta^{-1}\left(\frac{2}{2\sqrt{\zeta}\xi^2} + 1\right) = \frac{\frac{\sqrt{\xi^2}}{\sqrt{\pi}\sqrt{y}} - e^{\frac{y}{\xi^2}} \text{Erfc}\left(\sqrt{\frac{y}{\xi^2}}\right)}{4\xi^2}. \tag{5.12}$$

It remains to check that $h(y, \cdot)$ has an inverse Fourier transform with respect to the variable ξ . It is easily checked that, for all $y > 0$ fixed:

$$\begin{aligned} h(y, \xi) &= \mathcal{O}\left(\frac{\xi}{y^{3/2}}\right) \quad \text{as } \xi \rightarrow 0, \\ h(y, \xi) &= \frac{\frac{1}{\sqrt{\pi}}\sqrt{\frac{\xi^2}{y}} - 1}{4\xi^2} + \mathcal{O}\left(\frac{y}{\xi^2}\right) \quad \text{as } |\xi| \rightarrow +\infty. \end{aligned}$$

This function is then in $L^2(\mathbb{R})$ with respect to the ξ variable and has then an inverse Fourier transform with respect to ξ which is $g(y, x)$:

$$g(y, x) = \mathcal{F}_\xi^{-1}(h(y, \cdot))(x).$$

Moreover, for all $y > 0$, $g(y, \cdot) \in L^2(\mathbb{R})$ and the convolution of $g(y, \cdot)$ with itself is well defined

$$\mathcal{F}(g(y - y', \cdot) * g(y', \cdot))(\xi) = h(y - y', \xi)h(y', \xi)$$

and

$$\int_0^y \mathcal{F}(g(y - y', \cdot) * g(y', \cdot))(\xi) dy = \int_0^y h(y - y', \xi)h(y', \xi) dy.$$

Therefore,

$$\begin{aligned} & \int_{\mathbb{R}} |\mathcal{F}(g(y - y', \cdot) * g(y', \cdot))(\xi)| \, d\xi \\ & \leq \int_0^y \int_{\mathbb{R}} |h(y - y', \xi)h(y', \xi)| \, d\xi \, dy = \sum_{k=1}^6 I_k, \end{aligned}$$

with

$$\begin{aligned} I_1 & := \int_0^{y/2} \int_{|\xi| \leq y'^{1/2} \leq (y-y')^{1/2}} |h(y - y', \xi)h(y', \xi)| \, d\xi \, dy', \\ I_2 & := \int_0^{y/2} \int_{y'^{1/2} \leq |\xi| \leq (y-y')^{1/2}} |h(y - y', \xi)h(y', \xi)| \, d\xi \, dy', \\ I_3 & := \int_0^{y/2} \int_{y'^{1/2} \leq (y-y')^{1/2} \leq |\xi|} |h(y - y', \xi)h(y', \xi)| \, d\xi \, dy', \\ I_4 & := \int_{y/2}^y \int_{|\xi| \leq (y-y')^{1/2} \leq y'^{1/2}} |h(y - y', \xi)h(y', \xi)| \, d\xi \, dy', \\ I_5 & := \int_{y/2}^y \int_{(y-y')^{1/2} \leq |\xi| \leq y'^{1/2}} |h(y - y', \xi)h(y', \xi)| \, d\xi \, dy', \\ I_6 & := \int_{y/2}^y \int_{(y-y')^{1/2} \leq y'^{1/2} \leq |\xi|} |h(y - y', \xi)h(y', \xi)| \, d\xi \, dy'. \end{aligned}$$

We must verify that each term is finite. Indeed, we have

$$\begin{aligned} I_1 & \leq C \int_0^{y/2} \int_{|\xi| \leq y'^{1/2} \leq (y-y')^{1/2}} \frac{\xi^2}{y'^{3/2}(y - y')^{3/2}} \, d\xi \, dy' \\ & \leq C \int_0^{y/2} \frac{\min\{y'^{3/2}, (y - y')^{3/2}\}}{y'^{3/2}(y - y')^{3/2}} \, dy' < \infty; \\ I_2 & \leq C \int_0^{y/2} \int_{y'^{1/2} \leq |\xi| \leq (y-y')^{1/2}} \frac{|\xi|}{(y - y')^{3/2}} \\ & \quad \times \left(\frac{1}{\sqrt{y'}|\xi|} + \frac{1}{\xi^2} + \mathcal{O}\left(\frac{y'}{\xi^2}\right) \right) \, d\xi \, dy' \\ & \leq \frac{C}{y^{3/2}} \int_0^{y/2} \int_{y'^{1/2} \leq |\xi| \leq (y-y')^{1/2}} \left(\frac{1}{\sqrt{y'}} + \frac{1}{|\xi|} + 1 \right) \, d\xi \, dy \\ & \leq \frac{C}{y^{3/2}} \int_0^{y/2} \int_{y'^{1/2} \leq |\xi| \leq (y-y')^{1/2}} \left(\frac{2}{\sqrt{y'}} + 1 \right) \, d\xi \, dy < \infty; \\ I_3 & \leq C \int_0^{y/2} \int_{y'^{1/2} \leq (y-y')^{1/2} \leq |\xi|} \left(\frac{1}{\sqrt{y'}|\xi|} + \frac{1}{\xi^2} + \mathcal{O}\left(\frac{y'}{\xi^2}\right) \right) \\ & \quad \times \left(\frac{1}{\sqrt{y - y'}|\xi|} + \frac{1}{\xi^2} + \mathcal{O}\left(\frac{y - y'}{\xi^2}\right) \right) \, d\xi \, dy' \end{aligned}$$

$$\begin{aligned}
 &\leq C \int_0^{y/2} \int_{y^{1/2} \leq (y-y')^{1/2} \leq |\xi|} \left(\frac{1}{\sqrt{y'}\sqrt{y-y'}|\xi|^2} + \frac{1}{|\xi|^3} \left(\frac{1}{\sqrt{y-y'}} + \frac{1}{\sqrt{y'}} \right) \right. \\
 &\quad \left. + \frac{1}{\xi^4} + \mathcal{O}\left(\frac{y'}{\sqrt{y-y'}|\xi|^3}\right) + \mathcal{O}\left(\frac{y-y'}{\sqrt{y'}|\xi|^3}\right) + \mathcal{O}\left(\frac{y+y^2}{|\xi|^4}\right) \right) d\xi dy \\
 &\leq \frac{C}{\sqrt{y}} \int_0^{y/2} \frac{1}{\sqrt{y'}} \int_{y^{1/2} \leq (y-y')^{1/2} \leq |\xi|} \frac{d\xi}{|\xi|^2} dy \\
 &\quad + C \int_0^{y/2} \left(\frac{1}{\sqrt{y}} + \frac{1}{\sqrt{y'}} \right) \int_{y^{1/2} \leq (y-y')^{1/2} \leq |\xi|} \frac{d\xi}{|\xi|^3} dy \\
 &\quad + C \int_0^{y/2} \int_{y^{1/2} \leq (y-y')^{1/2} \leq |\xi|} \frac{d\xi}{|\xi|^4} dy \\
 &\leq \frac{C}{\sqrt{y}} \int_0^{y/2} \frac{dy}{\sqrt{y'}\sqrt{(y-y')}} + C \int_0^{y/2} \left(\frac{1}{\sqrt{y}} + \frac{1}{\sqrt{y'}} \right) \frac{dy}{y-y'} \\
 &\quad + C \int_0^{y/2} \frac{dy}{(y-y')^2}.
 \end{aligned}$$

Similar estimates show that the integrals I_4, I_5 and I_6 converge. The function $\int_0^y h(y - y', \xi)h(y', \xi) dy$ is then in $L^1(\mathbb{R})$ and has then an inverse Fourier transform which is

$$\int_0^y (g(y - y', \cdot) * g(y', \cdot))(\xi) dy.$$

In the second result of this section we obtain solutions of (5.1) for a particular class of initial data and we describe their long time asymptotic behavior in terms of self similar solutions (cf. Remark 5.5).

Theorem 5.4 *Suppose that the initial data $f_{in} \in L^1(Y; k_{\zeta}^2(y) dy)$ is even, non negative and such that*

$$\int_0^\infty \int_{\mathbb{R}^d} e^{s_0 m} f_{in}(m, p) dp dm < +\infty \tag{5.13}$$

for some $s_0 > 0$. Suppose also that the function F_0 defined as:

$$F_0(\zeta, \xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{\mathbb{R}^d} e^{-ip\xi} e^{-m\zeta} f_{in}(m, p) dp dm \tag{5.14}$$

satisfies:

$$\int_{\mathbb{R}^d} \int_{-\infty}^\infty |F_0(u + iv, \xi)| dv d\xi < \infty \tag{5.15}$$

when $u > s_0$. Then the function

$$f(t, m, p) = \mathcal{F}^{-1}(\mathcal{L}^{-1}F)(t, m, p) \tag{5.16}$$

$$F(t, \zeta, \xi) = \frac{H_0^2}{(H_0 + (t/2))^2 \left(\frac{1}{F_0(\xi, \xi)} - \frac{H_0 t/2}{H_0 + (t/2)} \right)}, \tag{5.17}$$

with $H_0 := M_{0,0}(0)^{-1}$ as defined in (2.6), is such that $f \in C^1((0, \infty); L^1(\mathbb{R}^+ \times \mathbb{R}^d))$ and satisfies (1.1), (1.2), (1.3), (1.4) with $a(y, y') \equiv 1$ for all $t > 0$ and almost every $y \in \mathbb{R}^+ \times \mathbb{R}^d$.

Furthermore, f satisfies

$$t^{(d+4)/2} f(t, tm, \sqrt{t}p) \rightarrow \varphi_\infty(m, p) := C_1 \frac{e^{-C_2 m}}{\sqrt{m}} \prod_{i=1}^d \sqrt{C_{3,i}} e^{-C_{3,i}^2 \frac{|p_i|^2}{m}}, \tag{5.18}$$

$$= C_1 \frac{e^{-C_2 m}}{\sqrt{m}} \left(\prod_{i=1}^d \sqrt{C_{3,i}} \right) e^{-\sum_{i=1}^d C_{3,i}^2 \frac{|p_i|^2}{m}} \tag{5.19}$$

in the weak sense of measures $\sigma(M^1(Y), C_c(Y))$, as $t \rightarrow +\infty$, where

$$C_1 = \frac{4}{(2\pi)^{d/2} M_{1,0}(f_{in})}, \quad C_2 = \frac{2}{M_{1,0}(f_{in})}, \tag{5.20}$$

$$C_{3,i} = \sqrt{\frac{2M_{1,0}(f_{in})}{M_{0,0}(f_{in})N_i(f_{in})}}, \quad N_i(f_{in}) = \int_0^\infty \int_{\mathbb{R}^d} p_i^2 f_{in}(m, p) dp dm. \tag{5.21}$$

Remark 5.5 Theorem 5.4 shows that the solutions of (5.1), for the set of initial data satisfying the hypothesis, behave asymptotically when $t \rightarrow +\infty$ as one of the self similar solutions obtained in Theorem 5.1. Notice indeed that (5.18) means that the solution $f(t, m, p)$ behaves, in a weak sense, as the self similar solution:

$$t^{-(d+2)/2} \varphi_\infty\left(\frac{m}{t}, \beta_1 \frac{p_1}{\sqrt{t}}, \dots, \beta_d \frac{p_d}{\sqrt{t}}\right)$$

as $t \rightarrow +\infty$. Notice also that this self similar solution is essentially the same as (5.11) in Remark 5.3. It is also noteworthy that, among all the possible self similar solutions, the solution f converges to one that is regular and non-negative.

Proof of Theorem 5.4 By Theorem 5.4, there exists a unique solution f of (5.1) in $C([0, T]; L^1(Y; k_S(y) dy)) \cap L^\infty(0, T; L^1(Y; k_S^2(y) dy))$ for all $T > 0$. Moreover, if the initial data is non-negative so is the solution for all time. It turns out that (5.1) may be explicitly solved using Fourier transform with respect to $p \in \mathbb{R}$ and Laplace transform with respect to $m > 0$. Let us then consider such a transform defined as:

$$F(t, \zeta, \xi) = \frac{1}{(2\pi)^{d/2}} \int_0^\infty \int_{\mathbb{R}^d} e^{-m\zeta} e^{-ip\xi} f(t, m, p) dp dm. \tag{5.22}$$

Due to the properties of the solution f we may then apply the transform (5.22) to both sides of (5.1) and obtain the following Bernoulli equation:

$$\partial_t F(t, \zeta, \xi) = \frac{1}{2} F^2(t, \zeta, \xi) - M_0(t) F(t, \zeta, \xi), \tag{5.23}$$

$$M_0(t) = F(t, 0, 0). \tag{5.24}$$

We first notice, taking $\zeta = \xi = 0$ in (5.23), that $M_0(t)$ satisfies $\frac{d}{dt}M_0(t) = -\frac{1}{2}M_0^2(t)$ from where

$$M_0(t) = \frac{1}{H_0 + t/2}. \tag{5.25}$$

The unique solution of (5.23) is the function $F(t, \zeta, \xi)$ given by (5.17). On the one hand, the function $t \mapsto (H_0t/2)/(H_0 + t/2)$ is strictly increasing with limit in infinity equal to H_0 , so that for any $\delta \in (0, 1)$ there exists $T \in (0, \infty)$

$$\forall t \in [0, T] \quad \frac{H_0t/2}{H_0 + t/2} \leq H_0(1 - \delta), \tag{5.26}$$

and on the other hand

$$|F(0, \zeta, \xi)| \leq \int_0^\infty \int_{\mathbb{R}} f(0, m, p) dm dp = H_0^{-1}. \tag{5.27}$$

Gathering (5.26) and (5.27) the fraction in the right hand side of (5.17) is well defined for all $t > 0$. More precisely for any $t \in [0, T]$ there is $\delta = \delta(t) > 0$ such that

$$\begin{aligned} \left| \frac{1}{F(0, \zeta, \xi)} - \frac{H_0t/2}{H_0 + (t/2)} \right| &\geq \left| \frac{1}{F(0, \zeta, \xi)} \right| - \frac{H_0t/2}{H_0 + (t/2)} \\ &\geq |F(0, \zeta, \xi)|^{-1} - H_0(1 - \delta) \\ &\geq \delta |F(0, \zeta, \xi)|^{-1}. \end{aligned} \tag{5.28}$$

By the hypothesis on f_{in} , for any fixed $\xi \in \mathbb{R}^d$ the function $F_0(\cdot, \xi)$ is analytic in the half plane $Re(\zeta) > -s_0$, it tends towards zero when in the half plane $Re(\zeta) \geq -s_0/2$, ζ tends two-dimensionally towards infinity. It immediately follows from (5.17) and (5.56) that $F(t, \cdot, \xi)$ satisfies the same properties for all $t > 0$ and $\xi \in \mathbb{R}^d$ as well as

$$\int_{\mathbb{R}^d} \int_{-\infty}^\infty |F(t, u + iv, \xi)| dv d\xi < \infty \tag{5.29}$$

for all $u > s_0$ for all $t > 0$. We deduce that for any $t > 0$ and almost every $\xi \in \mathbb{R}^d$

$$\int_{-\infty}^\infty |F(t, u + iv, \xi)| dv < \infty \tag{5.30}$$

and therefore the function $F(t, \cdot, \xi)$ is the Laplace transform of the function:

$$\mathcal{L}^{-1}(F(t, \cdot, \xi))(m) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{m\zeta} F(t, \zeta, \xi) d\zeta$$

the integral being independent of $x > s_0$. Notice that we have, for any $x > 0$:

$$\begin{aligned} &\frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \int_{\mathbb{R}^d} |e^{m\zeta} F(t, \zeta, \xi)| d\zeta d\xi \\ &= \frac{e^{xm}}{2\pi} \int_{-\infty}^\infty \int_{\mathbb{R}^d} |F(t, x + iv, \xi)| dv d\xi < \infty. \end{aligned}$$

The function $\xi \mapsto \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{m\zeta} F(t, \zeta, \xi) d\zeta$ is then integrable and we have:

$$\begin{aligned}
 f(t, m, p) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ip\xi} \mathcal{L}^{-1}(F(t, \cdot, \xi))(m) d\xi \\
 &= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \int_{\mathbb{R}^d} e^{ip\xi} e^{m\zeta} F(t, \zeta, \xi) d\zeta d\xi.
 \end{aligned}
 \tag{5.31}$$

In order to study the behavior of $f(t, m, p)$ as $t \rightarrow \infty$ it is a classical argument to consider the rescaled function φ associated to f by the relation

$$\varphi(t, M, P) := t^{(d+4)/2} f(t, tM, \sqrt{t}P),
 \tag{5.32}$$

so that

$$f(t, m, p) = t^{-(d+4)/2} \varphi\left(t, \frac{m}{t}, \frac{p}{\sqrt{t}}\right).
 \tag{5.33}$$

Taking the Fourier and Laplace transform in both side yields

$$F(t, \zeta, \xi) = t^{-1} \Phi(t, t\zeta, \sqrt{t}\xi)
 \tag{5.34}$$

with

$$\Phi(t, \zeta, \xi) = \frac{tH_0^2}{(H_0 + (t/2))^2 \left(\frac{1}{F(0, \frac{\zeta}{t}, \frac{\xi}{\sqrt{t}})} - \frac{H_0 t/2}{H_0 + (t/2)}\right)}.
 \tag{5.35}$$

Since we are interested in the long time behavior of $\Phi(\cdot, \zeta, \xi)$ for all ζ and ξ fixed, we may write:

$$\frac{1}{F(0, \frac{\zeta}{t}, \frac{\xi}{\sqrt{t}})} - \frac{H_0 t/2}{H_0 + (t/2)} = \frac{1}{F(0, \frac{\zeta}{t}, \frac{\xi}{\sqrt{t}})} - H_0 + \frac{H_0^2}{H_0 + (t/2)}$$

and consider the auxiliary function

$$\begin{aligned}
 \Psi(t, \zeta, \xi) &= \frac{tH_0^2}{((t/2))^2 \left(\frac{1}{F(0, \frac{\zeta}{t}, \frac{\xi}{\sqrt{t}})} - H_0 + \frac{H_0^2}{(t/2)}\right)} \\
 &= \frac{4H_0^2}{t \left(\frac{1}{F(0, \frac{\zeta}{t}, \frac{\xi}{\sqrt{t}})} - H_0 + \frac{2H_0^2}{t}\right)}.
 \end{aligned}
 \tag{5.36}$$

We perform the following expansion up to the order $o(1/t)$:

$$\begin{aligned}
 \frac{1}{F(0, \frac{\zeta}{t}, \frac{\xi}{\sqrt{t}})} - H_0 &= \frac{\zeta}{t} \frac{\partial F^{-1}}{\partial \zeta}(0, 0, 0) + \frac{\xi}{\sqrt{t}} \cdot \nabla_{\xi} F^{-1}(0, 0, 0) \\
 &\quad + \frac{1}{2t} \sum_{i,j=1}^d \xi_i \xi_j \frac{\partial^2 F^{-1}}{\partial \xi_i \partial \xi_j}(0, 0, 0) + o\left(\frac{1}{t}\right).
 \end{aligned}
 \tag{5.37}$$

Since by hypothesis f is even with respect to p , we have

$$\frac{\partial F}{\partial \xi_k}(0, 0, 0) = -i \int_0^\infty \int_{\mathbb{R}^d} f_{in}(m, p) p_k dp dm = 0$$

and then:

$$\frac{\partial F^{-1}}{\partial \xi_k}(0, 0, 0) = -\frac{1}{F(0, 0, 0)^2} \frac{\partial F}{\partial \xi_k}(0, 0, 0) = 0. \tag{5.38}$$

We also have

$$\frac{\partial^2 F}{\partial \xi_i \partial \xi_j}(0, 0, 0) = -\int_0^\infty \int_{\mathbb{R}^d} p_i p_j f(0, m, p) dp dm,$$

which with the help of (5.38) implies

$$\begin{aligned} \frac{\partial^2 F^{-1}}{\partial \xi_i \partial \xi_j}(0, 0, 0) &= -F^{-2}(0, 0, 0) \frac{\partial^2 F}{\partial \xi_i \partial \xi_j}(0, 0, 0) \\ &= H^2(0) \int_0^\infty \int_{\mathbb{R}^d} p_i p_j f(0, m, p) dp dm =: 2\mathcal{B}_{i,j}. \end{aligned} \tag{5.39}$$

Since $f_{in}(m, p)$ is even with respect to the p variable we have $\mathcal{B}_{i,j} = 0$ whenever $i \neq j$. Similarly, we compute

$$\frac{\partial F}{\partial \zeta}(0, 0, 0) = -\int_0^\infty \int_{\mathbb{R}^d} mf(0, m, p) dp dm,$$

which implies

$$\frac{\partial F^{-1}}{\partial \zeta}(0, 0, 0) = -\frac{1}{F^2(0, 0, 0)} \frac{\partial F}{\partial \zeta}(0, 0, 0) =: \mathcal{A}. \tag{5.40}$$

Thanks to (5.37), (5.38), (5.39) and (5.40), we deduce that (5.36) reads now:

$$\Psi(t, \zeta, \xi) = \frac{4H^2(0)}{(\zeta \mathcal{A} + \sum_i \mathcal{B}_i \xi_i^2 + 2H^2(0) + o(1))}$$

from where

$$\lim_{t \rightarrow +\infty} \Phi(t, \zeta, \xi) = \lim_{t \rightarrow +\infty} \Psi(t, \zeta, \xi) = \frac{4H^2(0)}{\mathcal{A}\zeta + \sum_i \mathcal{B}_i \xi_i^2 + 2H^2(0)} =: \Psi_\infty(\zeta, \xi). \tag{5.41}$$

In order to come back to the original variables, we recall that from standard integral calculus for any $\mathcal{C}, \mathcal{D} > 0$

$$\frac{1}{(2\pi)^{1/2}} \int_0^\infty \int_{\mathbb{R}^d} e^{-m\zeta} e^{-ip\xi} \frac{e^{-cm} e^{-\frac{1}{2}\mathcal{D}m^2}}{\sqrt{\mathcal{D}m}} dp dm = \frac{1}{\zeta + \mathcal{D}\xi^2 + \mathcal{C}},$$

from where choosing $\mathcal{C} := 2H_0^2/\mathcal{A}$ and $\mathcal{D}_i := \mathcal{B}_i/\mathcal{A}$, we obtain

$$\begin{aligned}
 (\mathcal{F}^{-1} \mathcal{L}^{-1})(\Psi_\infty) &= \frac{4H_0^2}{(2\pi m)^{1/2} \mathcal{A}} e^{-cm} \prod_{i=1}^d \frac{e^{-\frac{|p_i|^2}{2\mathcal{D}_i m}}}{\sqrt{\mathcal{D}_i}} \\
 &= \frac{4H_0^2 \mathcal{A}^{\frac{d}{2}-1}}{(2\pi m)^{1/2}} e^{-cm} \prod_{i=1}^d \frac{e^{-\frac{\mathcal{A}}{2m} |\frac{p_i}{\sqrt{\mathcal{B}_i}}|^2}}{\sqrt{\mathcal{B}_i}} = \varphi_\infty(m, p)
 \end{aligned}$$

as defined in (5.18). Finally, (5.65) implies that $\varphi(t, \cdot) \rightharpoonup \varphi_\infty$ in the weak sense of measures, which is nothing but (5.18). □

It is very easy to obtain a simplified version of Theorem 5.4 for (3.1)–(3.4) with $a(p, p') = 1$:

Theorem 5.6 *Suppose that the initial data $f_{in} \in M_{2\alpha}^1(\mathbb{R}^d)$, $\alpha \in \mathbb{N} \setminus \{0, 1\}$ is non negative and satisfies $f_{in}(p) = f_{in}(-p)$ and let F_0 be its Fourier transform:*

$$F_0(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{-ip\xi} f_{in}(p) dp. \tag{5.42}$$

Suppose moreover that

$$\int_{\mathbb{R}^d} |F_0(\xi)| d\xi < +\infty. \tag{5.43}$$

Then the function

$$f(t, p) = \mathcal{F}^{-1}(F)(t, p), \tag{5.44}$$

$$F(t, \xi) = \frac{H_0^2}{(H_0 + (t/2))^2 \left(\frac{1}{F_0(\xi)} - \frac{H_0 t/2}{H_0 + (t/2)} \right)}, \tag{5.45}$$

with $H_0^{-1} := F_0(0)$, is such that $f \in C([0, T]; M^1(\mathbb{R}^d) - weak) \cap L^\infty(0, T; M_{2\alpha}^1(\mathbb{R}^d))$, $f \in C([0, +\infty); L^\infty(\mathbb{R}^d)) \cap C([0, +\infty) \times \mathbb{R}^d)$ and satisfies (3.1)–(3.4) with $a(y, y') \equiv 1$ for all $t > 0$ and almost every $y \in \mathbb{R}^+ \times \mathbb{R}^d$.

Furthermore, f satisfies

$$t^{d/2} f(t, \sqrt{t} p) \rightharpoonup \varphi_\infty(p) \tag{5.46}$$

in the weak sense of measures $\sigma(M^1(Y), C_c(Y))$, as $t \rightarrow +\infty$ where

$$\varphi_\infty(p) = \mathcal{F}^{-1}(\Psi_\infty)(p), \tag{5.47}$$

$$\Psi_\infty(\xi) = \frac{4H^2(0)}{\sum_{i=1}^d \mathcal{B}_i \xi_i^2 + 2H^2(0)}, \tag{5.48}$$

$$\mathcal{B}_i = \int_{\mathbb{R}^d} p_i^2 f_{in}(p) dp. \tag{5.49}$$

In particular, when $d = 1$:

$$\varphi_\infty(p) = 2 \sqrt{\frac{2\pi}{M_2(f_{in})}} e^{-C_1|p|}. \tag{5.50}$$

Proof of Theorem 5.6 Equation (3.1)–(3.4) may be explicitly solved using Fourier transform with respect to $p \in \mathbb{R}$. By Theorem 3.1 we already know the existence and uniqueness of a solution $f \in C([0, T]; M^1(\mathbb{R}^d) - weak) \cap L^\infty(0, T; M^1_{2\alpha}(\mathbb{R}^d))$. Applying Fourier transform to both sides of (3.1)–(3.4) we obtain the following Bernoulli equation:

$$\partial_t F(t, \xi) = \frac{1}{2} F^2(t, \xi) - M_0(t) F(t, \xi), \tag{5.51}$$

$$M_0(t) = F(t, 0). \tag{5.52}$$

We first notice, taking $\xi = 0$ in (5.51), that $M_0(t)$ satisfies $\frac{d}{dt} M_0(t) = -\frac{1}{2} M_0^2(t)$ from where

$$M_0(t) = \frac{1}{H_0 + t/2}. \tag{5.53}$$

It is then straightforward to solve (5.51) and obtain the function $F(t, \xi)$ given by (5.45). On the one hand, the function $t \mapsto (H_0 t/2)/(H_0 + t/2)$ is strictly increasing with limit in infinity equal to H_0 , so that for any $\delta \in (0, 1)$ there exists $T \in (0, \infty)$

$$\forall t \in [0, T] \quad \frac{H_0 t/2}{H_0 + t/2} \leq H_0(1 - \delta), \tag{5.54}$$

and on the other hand

$$|F(0, \xi)| \leq \int_{\mathbb{R}^d} f(0, p) dp = H_0^{-1}. \tag{5.55}$$

Gathering (5.54) and (5.55) the fraction in the right hand side of (5.45) is well defined for all $t > 0$. More precisely for any $t \in [0, T]$ there is $\delta = \delta(t) > 0$ such that

$$\begin{aligned} \left| \frac{1}{F(0, \zeta, \xi)} - \frac{H_0 t/2}{H_0 + (t/2)} \right| &\geq \left| \frac{1}{F(0, \zeta, \xi)} \right| - \frac{H_0 t/2}{H_0 + (t/2)} \\ &\geq |F(0, \zeta, \xi)|^{-1} - H_0(1 - \delta) \\ &\geq \delta |F(0, \zeta, \xi)|^{-1}. \end{aligned} \tag{5.56}$$

Therefore:

$$|F(t, \xi)| \leq \frac{1}{\delta} |F_0(\xi)|$$

and (5.44) follows.

In order to study the behavior of $f(t, p)$ as $t \rightarrow \infty$ it is a classical argument to consider the rescaled function φ associated to f by the relation

$$\varphi(t, P) := t^{d/2} f(t, \sqrt{t} P), \tag{5.57}$$

so that

$$f(t, p) = t^{-d/2} \varphi\left(t, \frac{p}{\sqrt{t}}\right). \tag{5.58}$$

Taking the Fourier and Laplace transform in both side yields

$$F(t, \xi) = t^{-1} \Phi(t, \sqrt{t}\xi) \tag{5.59}$$

with

$$\Phi(t, \xi) = \frac{tH_0^2}{(H_0 + (t/2))^2 \left(\frac{1}{F(0, \frac{\xi}{\sqrt{t}})} - \frac{H_0 t/2}{H_0 + (t/2)} \right)}. \tag{5.60}$$

Since we are interested in the long time behavior of $\Phi(\cdot, \xi)$ for all ξ fixed , we may write:

$$\frac{1}{F(0, \frac{\xi}{\sqrt{t}})} - \frac{H_0 t/2}{H_0 + (t/2)} = \frac{1}{F(0, \frac{\xi}{\sqrt{t}})} - H_0 + \frac{H_0^2}{H_0 + (t/2)}$$

and consider the auxiliary function

$$\begin{aligned} \Psi(t, \xi) &= \frac{tH_0^2}{((t/2))^2 \left(\frac{1}{F(0, \frac{\xi}{\sqrt{t}})} - H_0 + \frac{H_0^2}{(t/2)} \right)} \\ &= \frac{4H_0^2}{t \left(\frac{1}{F(0, \frac{\xi}{\sqrt{t}})} - H_0 + \frac{2H_0^2}{t} \right)}. \end{aligned} \tag{5.61}$$

We perform the following expansion up to the order $o(1/t)$:

$$\frac{1}{F(0, \frac{\xi}{\sqrt{t}})} - H_0 = \frac{\xi}{\sqrt{t}} \cdot \nabla_{\xi} F^{-1}(0, 0) + \frac{1}{2t} \sum_{i,j=1}^d \xi_i \xi_j \frac{\partial^2 F^{-1}}{\partial \xi_i \partial \xi_j}(0, 0) + o\left(\frac{1}{t}\right). \tag{5.62}$$

Since by hypothesis f_{in} is even with respect to p , we have

$$\frac{\partial F}{\partial \xi}(0, 0) = -i \int_{\mathbb{R}^d} f_{in}(p) p \, dp = 0$$

and then:

$$\frac{\partial F^{-1}}{\partial \xi}(0, 0) = -\frac{1}{F(0, 0)^2} \frac{\partial F}{\partial \xi}(0, 0) = 0. \tag{5.63}$$

We also have

$$\frac{\partial^2 F^{-1}}{\partial \xi_i \partial \xi_j}(0, 0) = - \int_{\mathbb{R}^d} p_i p_j f(0, p) \, dp \, dm,$$

which with the help of (5.63) implies

$$\begin{aligned} \frac{\partial^2 F^{-1}}{\partial \xi_i \partial \xi_j}(0, 0) &= -F^{-2}(0, 0) \frac{\partial^2 F}{\partial \xi_i \partial \xi_j}(0, 0) \\ &= H^2(0) \int_{\mathbb{R}^d} p_i p_j f(0, p) \, dp =: 2\mathcal{B}_{i,j}. \end{aligned} \tag{5.64}$$

Since $f_{in}(m, p)$ is even with respect to the p variable we have $B_{i,j} = 0$ whenever $i \neq j$. We denote $B_{i,i} = B_i$. Thanks to (5.62), (5.63) and (5.64), we deduce that (5.61) reads now:

$$\Psi(t, \xi) = \frac{4H^2(0)}{(\sum_{i=1}^d B_i \xi_i^2 + 2H^2(0) + o(1))}$$

from where

$$\lim_{t \rightarrow +\infty} \Phi(t, \xi) = \lim_{t \rightarrow +\infty} \Psi(t, \xi) = \frac{4H^2(0)}{\sum_{i=1}^d B_i \xi_i^2 + 2H^2(0)} =: \Psi_\infty(\xi), \tag{5.65}$$

and (5.65) implies that $\varphi(t, \cdot) \rightharpoonup \varphi_\infty = \mathcal{F}^{-1}(\Psi_\infty)$ in the weak sense of measures, which is nothing but (5.46).

When $d = 1$, we recall from standard integral calculus that for any $C > 0$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ip\xi} \frac{d\xi}{1 + C\xi^2} = \sqrt{\frac{\pi}{2C}} e^{-| \frac{p}{\sqrt{C}} |},$$

from where for $C = B/2H_0^2$ we obtain

$$\varphi_\infty(p) = \mathcal{F}^{-1}(\Psi_\infty)(p) = 2\sqrt{\frac{\pi}{B}} H_0 e^{-| \frac{\sqrt{2}H_0}{\sqrt{B}} p |}$$

and (5.50) follows. □

Acknowledgements M.E. thanks support by Grant MTM2008-03541 from the Spanish Ministerio de Ciencia e Innovación and IT-305-07 from the Basque Government. S.M. thanks support by Grant MADCOF from the French Agence Nationale de la Recherche.

References

1. Bertoin, J.: Clustering statistics for sticky particles with Brownian initial velocity. *J. Math. Pures Appl.* **79**, 173–194 (2000)
2. Bertoin, J.: Self-attracting poisson clouds in an expanding universe. *Commun. Math. Phys.* **232**, 59–81 (2002)
3. Bobylev, A.V., Illner, R.: Collision integrals for attractive potentials. *J. Stat. Phys.* **95**, 633–649 (1999)
4. Brilliantov, N.V., Bodrova, A.S., Krapivsky, P.L.: A model of ballistic aggregation and fragmentation. *J. Stat. Mech.* P06011 (2009)
5. Brilliantov, N.V., Spahn, F.: Dust coagulation in equilibrium molecular gas. *Math. Comput. Simul.* **72**, 93–97 (2006)
6. Carnevale, G.F., Pomeau, Y., Young, W.R.: Statistics of Ballistic Agglomeration. *Phys. Rev. Lett.* **64**, 2913–2916 (1990)
7. Escobedo, M., Laurençot, Ph., Mischler, S.: On a kinetic equation for coalescing particles. *Commun. Math. Phys.* **246**, 237–267 (2004)
8. Escobedo, M., Mischler, S.: On a quantum Boltzmann equation for a gas of photons. *J. Math. Pures Appl.* **80**, 471–515 (2001)
9. Escobedo, M., Mischler, S.: On self-similarity for the coagulation equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23**, 331–362 (2006)
10. Escobedo, M., Mischler, S., Ricard, M.R.: On self-similarity and stationary problem for fragmentation and coagulation models. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22**, 99–125 (2005)
11. Estrada, P.R., Cuzzi, J.N.: Solving the coagulation equation by the moments method. *Astrophys. J.* **682**, 515–526 (2008)
12. Fournier, N., Laurençot, Ph.: Existence of self-similar solutions to Smoluchowski’s coagulation equation. *Commun. Math. Phys.* **256**, 589–609 (2005)

13. Fournier, N., Mischler, S.: A Boltzmann equation for elastic, inelastic, and coalescing collisions. *J. Math. Pures Appl.* **84**, 1173–1234 (2005)
14. Illner, R.: Stellar dynamics and plasma physics with corrected potentials: Vlasov, Manev, Boltzmann, Smoluchowski. In: *Hydrodynamic Limits and Related Topics*, Toronto, ON, 1998. *Fields Inst. Commun.*, vol. 27, pp. 95–108. *Am. Math. Soc.*, Providence (2000)
15. Jiang, Y., Leyvraz, F.: Scaling theory for ballistic aggregation. *J. Phys. A, Math. Gen.* **26**, L179–L186 (1993)
16. Krapivsky, P.L., Ben-Naim, E.: Aggregation with multiple conservation laws. *Phys. Rev. E* **53**, 291–298 (1996)
17. Laurençot, Ph., Mischler, S.: On coalescence equations and related models. In: *Modeling and Computational Methods for Kinetic Equations*. *Model. Simul. Sci. Eng. Technol.*, pp. 321–356. *Birkhauser Boston*, Boston (2004)
18. Leyvraz, F.: Scaling theory and exactly solved models in the kinetics of irreversible aggregation. *Phys. Rep.* **383**(2–3), 95–212 (2003)
19. Lynden-Bell, D.: Statistical mechanics of violent relaxation in stellar systems. *Mon. Not. R. Astron. Soc.* **136**, 101–121 (1967)
20. Mischler, S., Mouhot, C.: Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres. *Commun. Math. Phys.* **3**, 287 (2009)
21. Mischler, S., Wennberg, B.: On the spatially homogeneous Boltzmann equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16**, 467–501 (1999)
22. Norris, J.R.: Smoluchowski's coagulation equation: uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab.* **9**, 78–109 (1999)
23. Morfill, G., Röser, S., Tscharnuter, W., Völk, H.: The dynamics of dust in a collapsing protostellar cloud and its possible role in planet formation. *Moon Planets* **19**, 211–220 (1978)
24. Roquejoffre, J.M., Villedieu, Ph.: A kinetic model for droplet coalescence in dense sprays. *Math. Models Methods Appl. Sci.* **11**, 867–882 (2001)
25. Rudin, W.: *Functional Analysis*. McGraw-Hill, New York (1991)
26. Smoluchowski, M.: Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen. *Z. Phys. Chem.* **92**, 129–168 (1917)
27. Spahn, F., Albers, N., Sremčević, M., Thornton, C.: Kinetic description of coagulation and fragmentation in dilute granular particle ensembles. *Europhys. Lett.* **67**, 545–551 (2004)
28. Trizac, E., Hansen, J.P.: Dynamic scaling behavior of ballistic coalescence. *Phys. Rev. Lett.* **74**, 4114–4117 (1995)
29. Trizac, E., Krapivsky, P.L.: Correlations in ballistic processes. *Phys. Rev. Lett.* (2003). doi:[10.1103/PhysRevLett.91.218302](https://doi.org/10.1103/PhysRevLett.91.218302)
30. Wetherill, G.: *The Formation and Evolution of Planetary Systems*. Cambridge University Press, Cambridge (1988)
31. Williams, F.A.: Spray combustion and atomization. *Phys. Fluids* **1**, 541–545 (1958)
32. Wurm, G., Blum, J.: Experiments on preplanetary dust aggregation. *Icarus* **132**, 125–136 (1998)